Gustavo Curi Amarante

Changes in the Brazilian Yield Curve Response to Monetary Shocks

DISSERTAÇÃO DE MESTRADO

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Dissertação de Mestrado

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Prof. Carlos Viana de Carvalho
Advisor
Departamento de Economia — PUC–Rio

Prof. Tiago Berriel
Departamento de Economia – PUC–Rio

Prof. Marco Bonomo
Insper

Prof. Monica Herz
Coordenadora Setorial do Centro de Ciências Sociais – PUC–Rio

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Gustavo Curi Amarante

Graduated in Economics at Insper (São Paulo, Brazil) in 2012.

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Abstract

Amarante, Gustavo Curi; Carvalho, Carlos Viana de. **Changes in the Brazilian Yield Curve Response to Monetary Shocks**. Rio de Janeiro, 2015. 59p. Dissertação de Mestrado — Departamento de Economia, Pontifícia Universidade Católica do Rio de Janeiro.

Empirical evidence from reduced form VAR estimates shows that there has been a change in the way that the Brazilian yield curve reacts to a monetary policy shock. To better understand the sources of this change we estimated a linearized DSGE model with a term structure of interest rates over two sample periods to see what parameters of the economy might have caused the change. The linearization method is augmented with a risk adjustment term in order to generate a positive term spread and a risk-adjusted steady state. We discuss the empirical evidence, compare the solution methods with other traditional methods and estimate a model with Epstein-Zin preferences using Bayesian methods. We find that our structural model is capable of capturing some of the changes of behavior, and it is caused mainly by a smaller inflation coefficient of the interest rate rule and higher persistence of monetary policy shocks.

Keywords

Term structure of interest rates; Dynamic stochastic general equilibrium; Linearization method; Bayesian estimation;

Resumo

Amarante, Gustavo Curi; Carvalho, Carlos Viana de. **Changes in the Brazilian Yield Curve Response to Monetary Shocks**. Rio de Janeiro, 2015. 59p. Dissertação de Mestrado — Departamento de Economia, Pontifícia Universidade Católica do Rio de Janeiro.

Evidências empíricas de estimativas de modelos VAR em forma reduzida mostram que houve uma mudança na maneira que a curva de juros brasileira reage à choques de política monetária. Para melhor entender a razão desta mudança, estimamos um DSGE linearizado, acrescido de uma estrutura à termo para as taxas de juros, sobre dois períodos amostrais para verificar quais parâmetros da economia poderiam causado essa mudança. O método de linearização envolve um termo de ajuste que permite a existência de prêmio à termo e gera um estado estacionário ajustado pela volatilidade. Nós discutimos as evidências empíricas, comparamos o método de solução com outro métodos mais tradicionais e estimamos um modelo com preferências Epstein-Zin usando métodos bayesianos. Nós encontramos que nosso modelo estrutural é capaz de capturar algumas das mudanças de comportamento, que é causada principalmente por um menor coeficiente associado à inflação na regra de juros e por maior persistência dos choques monetários.

Palavras–chave

Estrutura à termo da taxa de juros; Equilíbrio geral dinâmico estocástico; Método de linearização; Estimação bayesiana;

Contents

Tho' much is taken, much abides; and though we are not now that strength which in old days moved earth and heaven; that which we are, we are; One equal temper of heroic hearts, made weak by time and fate, but strong in will to strive, to seek, to find, and not to yield.

Alfred, Lord Tennyson, *Ulysses.*

1
Introduction

Empirical evidence indicates that there may have been a change in the way that the Brazilian Central Bank conducts its monetary policy and this may have had an effect on the way the yield curve reacts to monetary policy shocks. To asses the question we first estimate reduced form VARs to evaluate the fully empirical relationship between the central bank target rate and its effect over the yield curve. We then investigate if a New Keynesian model augmented with Epstein-Zin preferences can shed some light on what structural changes might have caused this different behavior. To do so, we linearize the model using the risk adjusted method from Dew-Becker (2014), which the author calls "Essentially Affine Method", so that we can generate a non-zero term premium, even in a linearized environment. Since it results in a VAR solution for the macro variables and an affine solution for the term structure of interest rates, we can apply standard bayesian methods to estimate the model. We do this for two sample periods. The first one is from January 2004 until December 2010 and the second is from January 2011 until January 2015.

The time frames for these sample were chosen based on a change in the Brazilian political scenario. Ihara (2013) uses the synthetic control method and structural change tests to investigate a possible change in the way the central bank conducts monetary policy and finds two possible dates for structural breaks. The first one is in June 2003, possibly as a response to a confidence crisis due to the election of Luiz Inácio Lula da Silva, and is left out of the sample in this present work. The second structural change date he found is in February 2009, which is close to our chosen date and may also be related to changes in the Brazilian political cycle. Since our study uses a very different methodology, our sample time frame is chosen qualitatively based on the date that President Dilma Rousseff took office. We believe that small changes in the sample time frame will not affect the conclusion of this work.

In our approach, we want to estimate a structural model and see if the parameter estimates for each sample period can tell the story behind this change. Since we have reduced sample sizes for each period we need more observable variables in a our model in order to compensate for the lack of information, so we include all available maturities of the Brazilian yield curve. But including asset prices in a DSGE while maintaining tractability in order to use standard estimation methods is not an easy task. Research from Ang and Piazzesi (2003), Dewatcher and Lyrio (2006), Wu (2003) Shousha (2005) and

Rudebusch and Wu (2008), although having clear macroeconomic interpretation, rely on reduced form models for the dynamics of macro variables. The literature that tries to bring DSGE models and the yield curve together using structural models still has a few barriers, specially considering the estimation of these models.

Structural models with asset prices require risk-averse behavior of consumers. Mathematically, this is represented by the concavity of their utility function, but standard estimation methods require the linearization of such models which eliminate concavity and generate a risk-neutral behavior in the solution. This means that all assets would have the same expected return, independently of their risk. The proper way to solve models like these is to use higher-order solution methods. Hordhal, Tristani and Vestin (2007), Rudebusch and Swanson (2012) and Kung (2013) solve their models using second- and third- order approximations, but their analysis are based on calibration of the parameters and not estimation. Binsbergen et al (2012) estimates a third-order approximated model with a term structure of interest rates but they rely on methods that do not have well know properties and are computationally intensive.

Wu (2006) and Bekaert, Cho and Moreno (2010) try to include adjustments for risk to the model's linear solution in way that is not consistent across all equilibrium conditions of the model. There is a recent literature that tries to correct this problem. Dew-Becker (2012), Malkhozov and Shamloo (2010) and Meyer-Gohde (2014) developed similar methods that, although heavily dependent on functional forms of the model, can generate linear solutions that are very competitive with higher order solutions.

For our work, we use Dew-Becker's Essentially Affine Method to solve our model, which generates a risk-adjusted linear solution for the macroeconomic variables and an affine solution for the yield curve in the same form as Duffie and Kan (1996). These solutions allows us to cast our model in a state-space representation and estimate it using the kalman filter and standard bayesian methods. The model we estimate is a standard New Keynesian model with features that are important for the asset pricing literature, like the use of a time-varying inflation target and Epstein-Zin preferences with time-varying risk aversion. Since we are using the kalman filter, we can extract the smoothed non-observable variables, which is done for the inflation target. Our results point to an increase in the persistence of monetary policy shocks and smaller inflation coefficient in the interest rate rule of the central bank.

The remainder of this study is organized as follows: Section 2 we conduct a reduced form VAR study to asses the empirical relationship between the central banks target rate and the yield curve. This is done for other countries as well in order to see if a possible change in this relationship was exclusive to Brazil or some global factor that impacted several countries. Section 3 presents Dew-Becker's method for a general model and compares it to other solution methods by applying them to a small New Keynesian model. Section 4 presents the medium scale model, Dew-Becker's specific method for Epstein-Zin preferences, details of the estimation procedure and results. Section 5 concludes.

2
Vector Autoregressions

Following the same idea as Wu (2006), the objective of this section is to evaluate the empirical relationship between main macroeconomic variables and the term structure of interest rates and to see if there has been a change in the way that the Brazilian yield curve responds to monetary policy shocks. To see if such change is a fact exclusive to the Brazilian economy or if it is a general fact for other similar countries as well, we apply the same analysis for other emerging market inflation targeting countries. In this study we consider, Chile, Colombia, Mexico, South Africa and Turkey[1].

For each country, we estimated vector autoregressions (VAR) over two sample time frames. The first one is from January 2004 until December 2010, and the second one is from January 2011 until January 2015. We chose this date due to the end of a political cycle in the Brazilian government[2]. All data series have a monthly frequency. Our macroeconomic variables are industrial production, monthly inflation rate and the central bank's monetary policy rate. All the series were subjected to unit root tests[3] and the VAR lags were chosen parsimoniously based on several information criteria, given the reduced sample size. Shocks were identified using the Cholesky decomposition[4]. We also need to include information of the term structure of interest rates, but including all the different yields in the VAR would make the interpretation of the results very confusing and including just a few of them do not allow us to analyze the dynamics of the shape of the curve. So we summarized the information from the term structure of interest rates in the same way as Litterman and Scheinkman (1991) by using principal components analysis on the available series of yields up to 12-months of maturity and extract the first two components. Table 2.1 summarizes a few moments of the term structure for both sample periods and figures 2.1 through 2.6 show the factor loadings from the principal components

[1]Not all countries have data avaible for the first sample period. In that case, the country is only taken into account over the second sample period. Details on avaible data for each couontry are shown in the appendix.

[2]Robustness was checked by splitting the sample exactly in half, in which 17 observations of the first sample were moved to the second one and the break would be in April 209, a date much closer to the one found by Ihara (2013) . Results still hold.

[3]The only series that showed presence of a unit root were the mexican policy target rate and level factor over the second sample period. This country was not included in the analysis over the second sample period.

[4]The ordering of the variables is: industrial production, inflation, policy rate, level factor, slope factor. This means that the level and slope factors are allowed to have a simultaneous response to monetary policy shocks while industrial production and inflation are not.

Table 2.1: Empirical moments of the Brazilian term structure

Variable	Mean	Std. Dev.	Autocorr.(1)	Autocorr.(12)
Sample 1: Jan. 2004 - Dec. 2010				
1-month rate	0.1345	0.0336	0.986	0.482
6-month rate	0.1352	0.0323	0.982	0.449
12-month rate	0.1369	0.0298	0.972	0.401
Level Factor	0	0.1054	0.983	0.458
Slope Factor	0	0.0144	0.898	0.008
Sample 2: Jan. 2011 - Jan. 2015				
1-month rate	0.0980	0.0178	0.962	-0.220
6-month rate	0.0995	0.0184	0.945	-0.169
12-month rate	0.1014	0.0188	0.937	-0.146
Level Factor	0	0.0606	0.948	-0.173
Slope Factor	0	0.0076	0.871	-0.138

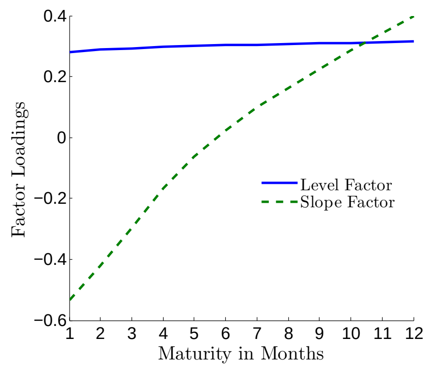

Figure 2.1: Factor loadings for Brazil

for each country.

Alves et al (2011) present some stylized facts from the Brazilian yield curve. Their sample period coincides with the first sample period of this work. Some of the facts that the authors point out are that the dynamics of each yield are very persistent and that the shorter ends of the curve are more volatile than the longer ones. This can also be seen in table 2.1 for the first sample period. But this fact changes over the second sample period, in which the longer ends of the curve show a higher volatility and the persistence, although still strong for a short period of time, dies out much faster for longer autocorrelation lags.

Following the nomenclature from principal components literature, the first component is labeled "Level Factor", since the factor loadings for all maturities are around the same magnitude and the factor is highly correlated with the average level of the yield curve, and the second component is labeled

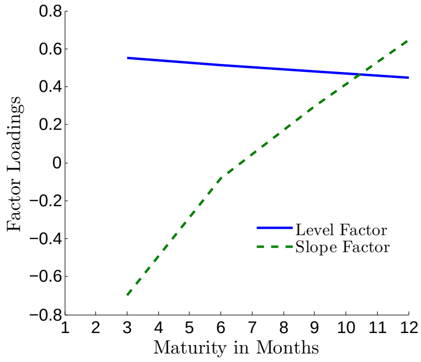

Figure 2.2: Factor loadings for Chile

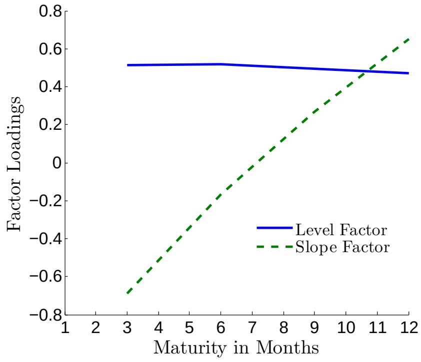

Figure 2.3: Factor loadings for Colombia

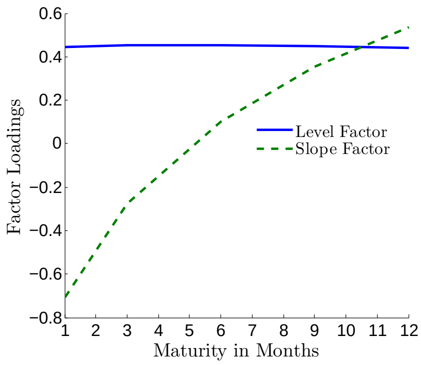

Figure 2.4: Factor loadings for Mexico

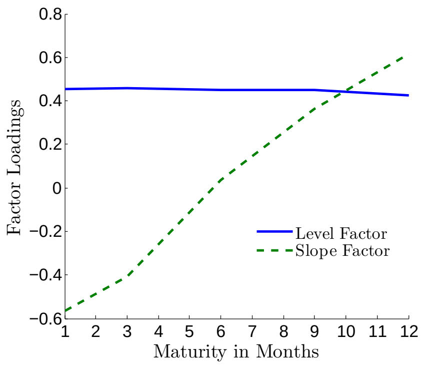

Figure 2.5: Factor loadings for South Africa

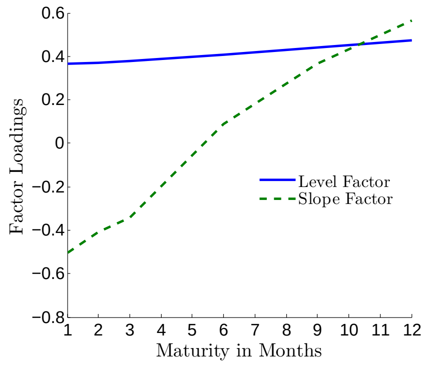

Figure 2.6: Factor loadings for Turkey

"Slope Factor", since the yields from shorter ends have a negative impact on the factor and the yields from longer ends have a positive impact, and it is highly correlated with the spread between the long and short end of the yield curve. For all countries, the first two components explain more than 99% of the total variance of their term structure.

The first three lines of graphs of figure 2.7 shows the Brazilian impulse response functions (IRF) of macroeconomic variables to a 50 basis points increase in the monetary policy rate and their 95% confidence interval for both sample periods. A contractionary monetary policy shock has a significant but transitory effect on the central bank's target rate, lasting for about 14 months over the first sample period and about 11 months on the second one. This is a sign of persistence in the way the central bank sets the target rate. This shock is not enough to generate a significant effect over the industrial production over both sample periods. The same happens for inflation. A 50 basis point increase has no significant effect over it, however, due to the sign of the impulse response function, we might find evidence that inflation would rise as a response to a contractionary monetary policy shock, indicating a "price puzzle" as described in Sims (1992).

The last two lines of figure 2.7 shows the impulse response functions of the first two principal components of the Brazilian yield curve. We would expect for monetary policy shocks to have a smaller and shorter-lived effect for longer maturities. In other words, we would expect and upward shift in the general level of the curve while making it flatter. Over the first sample period we find this movement on the curve and although the effect of the shock on the slope is small, it is still significant for more than a year. This same movement is found by Wu (2006) for the american yield curve. However, on the second sample period the shock has the same initial impact over the level of the curve but, due to a hump-shaped response, the effect reaches a peak around 7 months after the shock. It also generates an initial positive impact

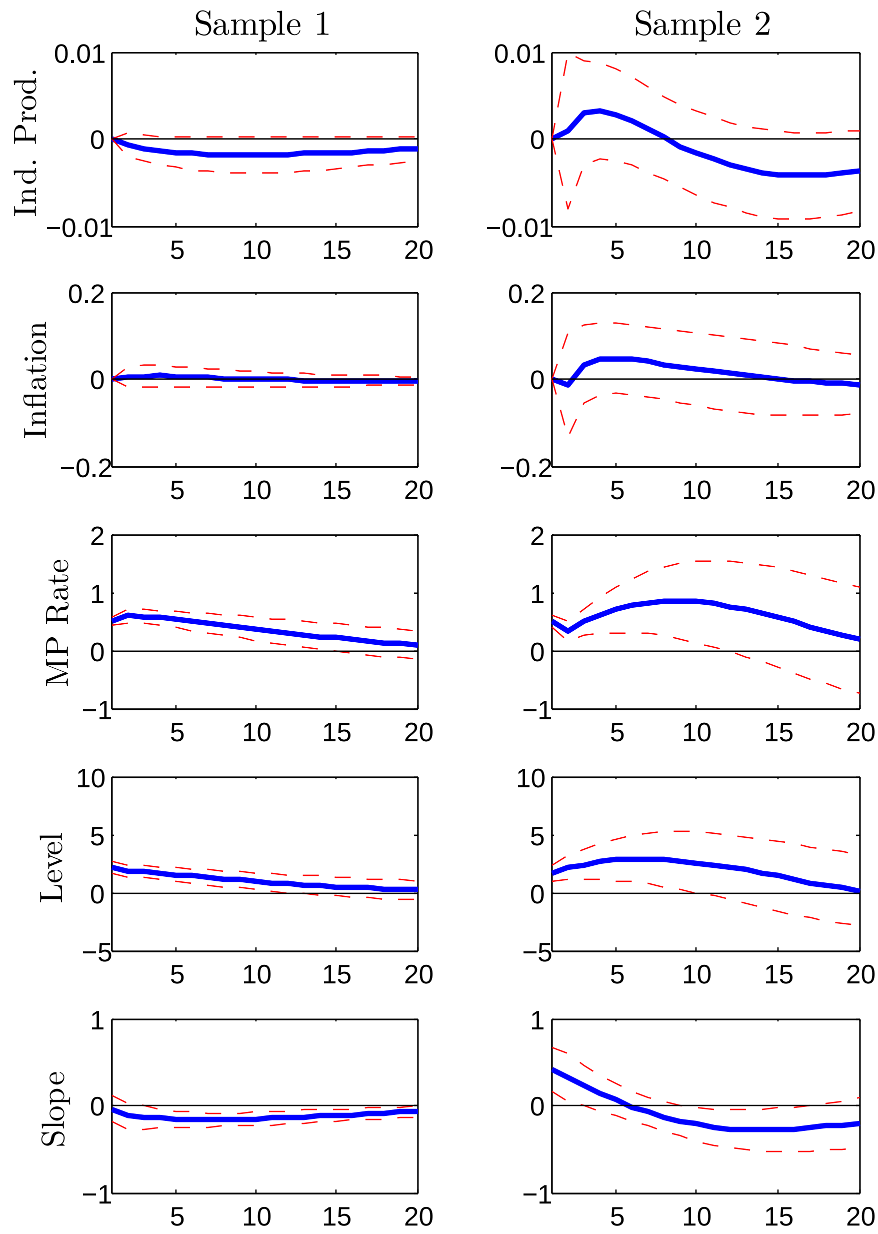

Figure 2.7: Impulse response functions of a 50bps monetary policy rate increase for Brazil

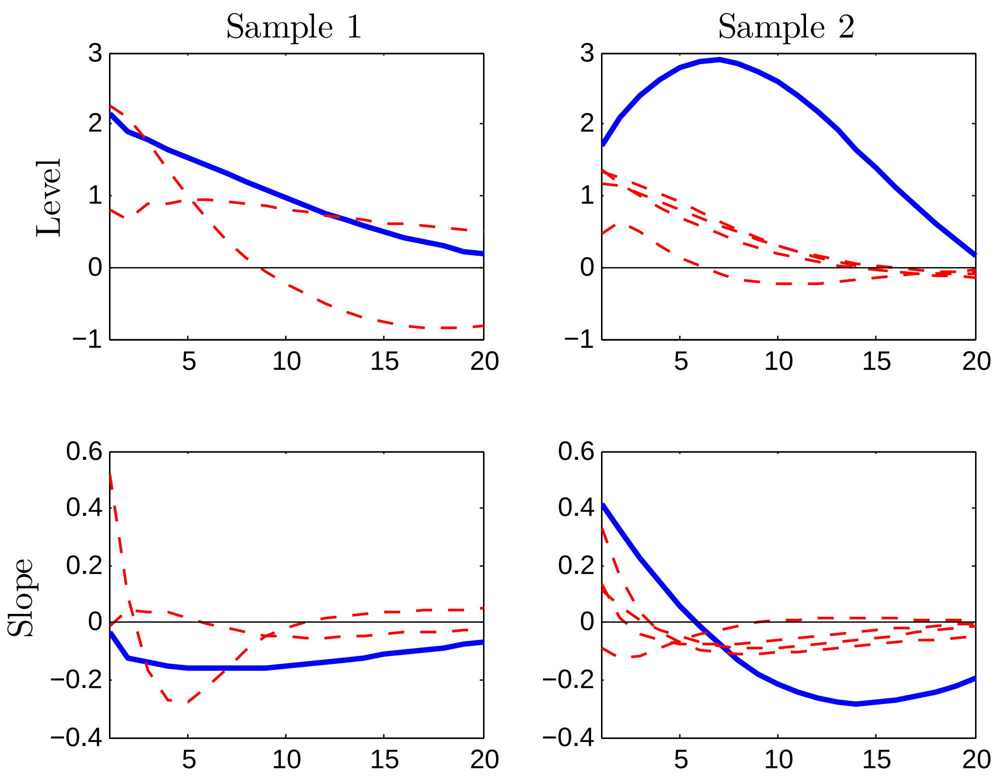

Figure 2.8: IRFs of a monetary policy shock for yield curve factors for both samples periods. Brazil is the solid line and other countries are the dashed ones. On the first sample the countries are Brazil, Mexico and South Africa. On the second sample period there are also Chile, Colombia and Turkey.

on the slope, meaning that longer ends of the curve would initially suffer a higher impact than the shorter ends, which goes against the stylized facts, so now the curve would be steeper than it was before the shock, but this effect dissipates after 6 months and the curve turns flatter than before the shock.

As mentioned before, we may also suspect that this change in the way the yield curve responds to monetary policy shocks is not exclusive to Brazil. To investigate this hypothesis, we estimated the same VAR for other countries. Figure 2.8 shows the impulse response functions for the level and slope factors for all analyzed countries over both sample periods. Given a 50 basis point increase in their monetary policy rates, we can see that in the first period all countries have a positive and persistent effect over the level of the yield curve, while the effect over the slope factor has different directions and intensities for each country. On the second sample period we have that the effect of the monetary policy shock over the level of the yield curve is a lot more persistent in Brazil than in other countries, even generating a hump-shaped response, while the effect over the slope of the curve is initially positive and stronger than it was before and is now a lot more persistent than the response of other countries. Of course that these countries, although comparable, have very different behaviors after a monetary policy shock, but the change for the case of Brazil is especially accentuated. This indicates that the change in the way that the yield curve reacts to a shock must come from a specific characteristic of Brazil, and not due to a global common factor.

Given this change in the way that the Brazilian term structure reacts to monetary policy shocks we want a theoretical reasoning and empirical investigation of possible reasons for this change in impact of monetary policy shocks. In other words, we are looking for a structural model capable of capturing this change while maintaining tractability and good well know properties.

3
Comparing Solution Methods

The equilibrium conditions of DSGE models are a non-linear rational expectations system of equations. Finding a closed form analytical solutions to these models is, most of the time, impossible and numerical solutions can be computationally intensive. That is why is common to apply some kind of approximation to the system before solving it. But the models in this study have to deal with bond pricing, which surely involves volatility and risk aversion which are usually characterized by second- or higher-order moments. Risk aversion is mathematically represented by the concavity of the household's utility function, but in the structural macroeconomic literature it is common to do a first order approximation of the model around its non-stochastic steady state, which eliminates the concavity creating a "certainty equivalent" behavior in the solution. In an asset pricing environment this means that every asset would have the same expected return and the steady-state yield curve would be completely flat.

Second- or higher-order approximations can capture the concavity of the model but estimating these models require computationally intensive methods. Hördhal, Tristani e Vestin (2007) do a second-order approximation to a new Keynesian model but resort to calibration instead of estimation. A solution to a second order approximation can be computed with perturbation methods in less than a second, but evaluating its likelihood function takes more computationally intensive methods. Rudebusch and Swanson (2012) solve their model with a third order approximation. They say that a perturbation method solution can take a few minutes to be computed. Bisbergen et al (2012) solve a DSGE with a third-order approximation. They use a particle filter to evaluate the likelihood function of the model. Finding the maximum likelihood estimator is complicated since the likelihood function of higher-order models may be rugged and multimodal and since the particle filter makes the likelihood function not differentiable with respect to the model parameters, this method requires unusual evolutionary algorithms for optimization. All of this is clearly computationally expensive.

Due to these problems, there is a growing literature that tries to linearize DSGE models while making risk adjustments in order to allow for estimation with traditional methods and include asset pricing. The main idea behind these methods is to linearize the model around a "stochastic steady-state", which can be defined as fixed point of a system of rational expectations equations where there is risk, meaning that exogenous shocks have a known distribution, but all the realization of the shocks are set to zero, despite the model having been solved under the assumption of a non-degenerate distribution for the shocks.

The "first generation" of this risk adjusted method can be well characterized by Wu (2006) and Bekaert, Cho e Moreno (2010). Both solve a new Keynesian model with first-order perturbation methods and substitute the solution in the equation of the stochastic discount factor, which in turn is log-linearized under the assumption that the variables are log-normally distributed. This means that there are two different linearization methods involved in the final solution. The model dynamics, the steady-state of the macro variables and the magnitude of the impact of shocks are still the same of traditional first-order perturbation methods, but it changes the steady-state values for the bond yields due to the introduction of a constant term in the yield equations. This extra constant term appears because of the expectation operator in the stochastic discount factor equation and the hypothesis that it is log-normally distributed. The expectation operator represents the household's choice under uncertainty, but in a large model there might be other equations that also involve the expectations operator, like the price setting decision of firms, and the risk behind these other uncertainties are being ignored by their solution methods.

The "second generation" of the risk-adjusted solution methods apply the same approximation methods to all the equations in the system which generates a better approximation around the stochastic steady-state since all equations are allowed to have a risk adjustment and not only the ones related to asset pricing. Malkhozov and Shamloo (2010) show that, assuming that all variables in the system are log-normally distributed, there is a linear solution where the coefficients generated by their risk adjusted method are very similar to ones from a second-order perturbation method. Meyer-Gohde (2014) also develops a very similar method and approximates the model around the stochastic steady-state and around the ergodic mean of the system which allows him to estimate his model with traditional bayesian methods. Dew-Becker (2012) also develops an algorithm for a risk-adjusted linear solution, which was chosen for this work and will be explained in details later. The cost of using these methods is that they rely heavily on assumptions over the distributions of the variables and functional forms, specially of the household's utility function, which is key for the stochastic discount factor.

The objective of this section is to compare and better understand the impact of each solution method over moments, steady-state values and impulse response functions of the solutions. We present a small new Keynesian model, which we follow closely from Galí (2008), that we solve using two risk-adjusted solution methods, one from the "first generation" and one from the second generation. Respectively, Wu's (2006) two-stage strategy and Dew-Becker's (2012) general case[1] for the Essentially Affine Approach. We then present a baseline calibration of the model and compare IRFs and steady state-values for both risk-adjusted solution methods and also for other traditional methods. We also show how different calibrations of interest rate persistence affect the shape of the IRFs.

3.1 Small New Keynesian Model

3.1.1 Household

There is a representative household that maximizes his lifetime utility function, which is separable in consumption C_t and labor N_t:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi} \right\}$$

where β is the discount factor, σ is the intertemporal elasticity of substitution and the relative risk aversion coefficient and φ is the Frisch elasticity of labor. C_t is the amount of final goods consumed and N_t is the number of hours worked. Households can also buy one-period bonds B_t, at price $Q_{t,t+1}$, that pays one nominal unit at maturity. Soon we will add other zero-coupon bonds to the system, but from Sargent & Ljungvist (2012) we know that this will not change the optimal allocations of the other variables of the economy since these additional bonds will be redundant assets in an economy with with complete markets. The household's budget constraint is given by:

$$P_t C_t + Q_{t,t+1} B_t \leq B_{t-1} + W_t^n N_t + T_t$$

where P_t is the aggregate price index, W_t^n is the nominal wage and T_t is a lump-sum component of income, which may include profits from ownership of firms. The solution to the household's problem yields an Euler equation for consumption and a labor supply. These are given by:

$$Q_{t,t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}}$$

[1]On the next session, I present the author's modification of the method for the specific case of models with Epstein-Zin (1989) preferences with time-varying risk aversion, since the modification depends on the functional form of the model.

$$\frac{W_t^n}{P_t} = C_t^{\sigma} N_t^{\varphi}$$

From this point on, $W_t = \frac{W_t^n}{P_t}$ denotes the real wage.

3.1.2 Final Goods Producer

The final good producer is competitive in both input and output markets and has a CES production function:

$$Y_t = \left(\int_0^1 Y_t(i)^{\frac{\varepsilon-1}{\varepsilon}} \, di \right)^{\frac{\varepsilon}{\varepsilon-1}}$$

where $Y_t(i)$ is the amount of the intermediate good bought from the intermediate producer i, ε is the elasticity of technical substitution between the inputs. The profit maximization problem yields a demand curve for every variety of intermediate good i and a price index, which are respectively given by:

$$Y_t(i) = \left(\frac{P_t(i)}{P_t} \right)^{-\varepsilon} Y_t$$

$$P_t = \left(\int_0^1 P_t(i)^{1-\varepsilon} \, di \right)^{\frac{1}{1-\varepsilon}}$$

3.1.3 Intermediate Goods Producer

There is a continuum of intermediate goods producers, indexed by $i \in [0,1]$, operating in monopolistic competition. Each i-th producer has a production function for the variety i given by:

$$Y_t(i) = A_t N_t(i)$$

where $N_t(i)$ is the amount of labor hired by the i-th producer and A_t is a technology term, common to all firms in the economy, that follows an AR(1) process given by:

$$\ln(A_t) = \rho_a \ln(A_{t-1}) + \sigma_a \varepsilon_t^a$$

Intermediate producers solve two problems: choosing how much labour to hire (sourcing problem) and setting their selling price (pricing problem). The sourcing problem yields the efficient real marginal cost for the firms:

$$MC_t = \frac{W_t}{A_t}$$

Note that the marginal cost is the same for all of the intermediate producers since it does not depend on i. Intermediate producers operate under the Calvo (1983) pricing scheme, where in each period there is a fraction $1-\theta$ of firms that can reset their prices. This optimal pricing of the firm consists in maximizing the discounted future profits by setting the firms current price, subject to the constraint of his own demand. The problem is given by:

$$\max_{P_t^\star} \quad E_t \sum_{k=0}^{\infty} \theta^k Q_{t,t+k} \left\{ \left(P_t^\star - P_{t+k} MC_{t+k} \right) Y_{t+k}(i) \right\}$$

$$s.t. \qquad Y_{t+k}(i) = \left(\frac{P_t^\star}{P_{t+k}} \right)^{-\varepsilon} Y_{t+k}$$

The first order condition of this problem is an infinite sum. With the help of two auxiliary variables, $G_{1,t}$ and $G_{2,t}$, it is possible to write this condition in recursive form like:

$$\frac{P_t^\star}{P_t} G_{1,t} = \frac{\varepsilon}{\varepsilon - 1} G_{2,t}$$

$$G_{1,t} = Y_t + \theta E_t \left\{ Q_{t,t+1} \Pi_{t,t+1}^{\varepsilon} G1, t+1 \right\}$$

$$G_{2,t} = Y_t MC_t + \theta E_t \left\{ Q_{t,t+1} \Pi_{t,t+1}^{1+\varepsilon} G_{2,t+1} \right\}$$

3.1.4 Central Bank

The central bank follows the interest rate rule:

$$R_t = R_{t-1}^{\rho_R} \left[\bar{R} \Pi_t^{\rho_\pi} \left(\frac{Y_t}{Y_t^n} \right)^{\rho_y} \right]^{1-\rho_R} \exp\left(v_t \right)$$

where Y_t^n is the natural output, $\bar{R}$ is the steady-state nominal interest rate and v_t is a monetary policy shock that follows an AR(1) process:

$$v_t = \rho_v v_{t-1} + \sigma_v \varepsilon_t^v$$

3.1.5 Market Clearing and Aggrregation

Under Calvo pricing scheme, it can be shown that the aggregate price index dynamics is given by:

$$1 = \theta \Pi_t^{\varepsilon - 1} + (1 - \theta) \left(\frac{P_t^\star}{P_t} \right)^{1-\varepsilon}$$

Clearing in the labor and final goods markets implies that:

$$N_t = \int_0^1 N_t\,(i)\,di$$

$$Y_t = C_t$$

With this it is possible to show that the aggregate production function is given by:

$$Y_t = \frac{A_t}{S_t} N_t$$

where S_t is a term associated with price dispersion, the inefficiency generated by price rigidities. Up to a first order approximation, S_t has no impact on the other variables, but it will be useful for us to have it explicitly in the aggregate production function for when we are doing second order approximations. Due to a Calvo pricing scheme it can be shown that:

$$S_t = (1-\theta)\left(\frac{P_t^\star}{P_t}\right)^{-\varepsilon} + \theta \Pi_t^{\varepsilon} S_{t-1}$$

We can also find the aggregate production function that would hold under flexible prices to find the natural output:

$$Y_t^n = \left[\left(\frac{\varepsilon - 1}{\varepsilon}\right) A_t^{1+\varphi}\right]^{\frac{1}{\sigma+\varphi}}$$

3.1.6 Term Structure of Interest Rates

We know from Ljungvist & Sargent (2012) that sequential trading of one-period bonds are enough characterize the equilibrium of the system and that bonds with longer maturities are redundant assets and can be priced using a no-arbitrage condition. Using this condition, the pricing of zero-coupon bonds is given by:

$$b_{n,t} = E_t\left[Q_{t,t+1} b_{n-1,t+1}\right]$$

where $b_{n,t}$ is the time t price of the bond that matures in n periods and pays 1 nominal unit at maturity, and $Q_{t,t+1}$ is the nominal stochastic discount factor. The time t continuously compounded yield to maturity of a n-period bond is given by:

$$r_{n,t} = -\frac{\ln\left(b_{n,t}\right)}{n}$$

3.2 Wu's Method

Wu (2006) solves his model with a two-step strategy. In the first step, he log-linearizes the equilibrium conditions of the model and solves the resulting linear system using traditional methods. In the second step, he substitutes the linear solution of the macro variables in the stochastic discount factor and uses a log-normal approximation only on this bond-pricing relevant Euler equation. As discussed before, this method contains problems. But since our objective here is only to understand how volatility enters the system and affects bond yields, we can use Wu's solution method, since it brings the advantage of having an affine analytical solution for bond yields.

We can make enough simplifications to the small New Keynesian model to get an analytical solution. We consider the model described in the previous subsection where the central bank does not reacts to the deviations of output from its natural level ($\rho_y = 0$). Following the first step of his method, we can log-linearize all the equations and make substitutions to obtain the system:

$$y_t = \vartheta_y^n + \psi_{ya}^n a_t + \tilde{y}_t$$

$$\tilde{y}_t = E_t\left(\tilde{y}_{t+1}\right) - \frac{1}{\sigma}\left(\hat{i}_t - E_t\left(\pi_{t+1}\right) + \left(1-\rho_a\right)\sigma\psi_{ya}^n a_t\right)$$

$$\pi_t = \beta E_t\left(\pi_{t+1}\right) + \kappa\tilde{y}_t$$

$$\hat{i}_t = \rho_i \hat{i}_{t-1} + \left(1-\rho_i\right)\rho_\pi \pi_t + v_t$$

$$a_t = \rho_a a_{t-1} + \sigma_a \varepsilon_t^a$$

$$v_t = \rho_v v_{t-1} + \sigma_v \varepsilon_t^v$$

In this very small model, $\hat{i}_t$ denotes the monetary policy rate as a deviation from its steady state value, y_t is the log of the level of output, $\tilde{y}_t$ is the output gap, π_t is the inflation rate, a_t is a productivity shock and v_t is a monetary policy shock. With some substitutions and manipulations we can find an analytical solution of the system in the form:

$$\tilde{y}_t = \lambda_{yi}\hat{i}_{t-1} + \lambda_{ya}a_t + \lambda_{yv}v_t$$

$$\pi_t = \lambda_{\pi i}\hat{i}_{t-1} + \lambda_{\pi a}a_t + \lambda_{\pi v}v_t$$

Now for the second step of the author's solution method, the pricing equation is given by:

$$b_{n,t} = E_t\left(Q_{t,t+1}b_{n-1,t+1}\right)$$

Under the hypothesis that $b_{n,t}$ and $Q_{t,t+1}$ follow a joint log-normal distribution:

$$\ln\left(b_{n,t}\right) = E_t\left[q_{t,t+1} + \ln\left(b_{n-1,t+1}\right)\right] + \frac{1}{2}Var_t\left[q_{t,t+1} + \ln\left(b_{n-1,t+1}\right)\right]$$

The log of the household's stochastic discount factor is given by:

$$q_{t,t+1} = \ln\left(\beta\right) - \sigma y_{t+1} + \sigma y_t - \pi_{t+1}$$

Substituting the solution of the macro variables in the log of the stochastic discount factor and, in turn, substituting this last one in the log of the pricing equation, it is possible to show that for a n-period bond, we have an affine solution:

$$r_{n,t} = -\frac{1}{n}\left(A_n + B_{i,n}\hat{i}_{t-1} + B_{a,n}a_t + B_{v,n}v_t\right)$$

where the coefficients for each maturity are given by:

$$A_n = A_{n-1} - \rho + \frac{1}{2}\left\{\left[B_{n-1,a} - \sigma\left(\psi_{ya}^n + \lambda_{ya}\right) - \lambda_{\pi a}\right]^2\sigma_a^2 + \left[B_{n-1,v} - \sigma\lambda_{yv} - \lambda_{\pi v}\right]^2\sigma_v^2\right\}$$

$$B_{i,n} = \left(B_{i,n-1} - \sigma\lambda_{yi} - \lambda_{\pi i}\right)\left(\phi_i + \phi_\pi\lambda_{\pi i}\right) + \sigma\lambda_{yi}$$

$$B_{a,n} = B_{a,n-1}\rho_a + \sigma\left(\psi_{ya}^n + \lambda_{ya}\right)\left(1-\rho_a\right) - \rho_a\lambda_{\pi a} + \left(B_{i,n-1} - \sigma\lambda_{yi} - \lambda_{\pi i}\right)\phi_\pi\lambda_{\pi a}$$

$$B_{v,n} = B_{v,n-1}\rho_v + \sigma\lambda_{yv}\left(1-\rho_v\right) - \rho_v\lambda_{\pi v} + \left(B_{i,n-1} - \sigma\lambda_{yi} - \lambda_{\pi i}\right)\left(1 + \phi_\pi\lambda_{\pi v}\right)$$

Notice that if there were no risk adjustment, the steady state of the curve would be $\rho = -\ln\left(\beta\right)$ for all maturities, so that all bonds would have the same expected return. With the risk adjustment we can see that the volatilities of the shocks are a component of the constant term for each rate. An increase in the volatility of shocks lowers the value of the intercept of the yield equations, which means they have a negative effect on the steady-state values of the curve. The higher the volatility of shocks, the smaller the steady state yields. This happens because in a scenario where you have more uncertainty (higher volatilities of the shocks) you have a smaller expected utility, so an asset that pays nominal consumption units should have a higher price, since it works like an insurance on consumption. The higher the price of a bond, the smaller the yield towards its face value. We can also see that the effect of the shocks on longer maturities are amplified when the shocks are more persistent since its effect would be present in the economy for a longer period. But notice that the constant terms only change the steady-state value of the yields, having no effect over the other variables in the model. In other words, there is no precautionary savings effect over consumption when the uncertainty rises.

3.3 Essentially Affine Method - General Case

Dew-Becker (2012) explains three different ways to apply his method. The first one is for a general case, which is explained in this section, the second takes advantage of models with Epstein-Zin preferences and time-varying risk aversion, explained in a later section, and a third one for models with stochastic volatility.

For a general case, we are going to represent all of the variables of the system by the vector X_t, which has dimensions $N_X \times 1$. So the whole system can be represented as:

$$G\left(X_t, X_{t+1}, \varepsilon_{t+1}\right) = 0_{N_D \times 1}$$

The non-stochastic steady-state of the model is defined by the vector $\bar{X}$ that solves:

$$G\left(\bar{X}, \bar{X}, 0\right) = 0_{N_D \times 1}$$

We can then separate the equations of the system into two sets. The first is a set of N_D non-expectational equations (like budget constraints and exogenous processes). The second set includes the N_F equations that involve the expectations operator, where $N_X = N_D + N_F$.

$$G\left(X_t, X_{t+1}, \varepsilon_{t+1}\right) = \begin{pmatrix} D\left(X_t, X_{t+1}, \varepsilon_{t+1}\right) \\ E_t\left[M\left(X_t, X_{t+1}, \varepsilon_{t+1}\right) \times F\left(X_t, X_{t+1}, \varepsilon_{t+1}\right)\right] - 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

D and F are vector-valued functions and M is a scalar valued function that represents a stochastic discount factor or pricing kernel. The author of the method points out that this formulation does not restrict function F, so if we have a set of equilibrium conditions $E_t\left[J\left(X_t, X_{t+1}, \varepsilon_{t+1}\right)\right] = 1$ that do not explicitly involve the stochastic discount factor. We can define $F\left(X_t, X_{t+1}, \varepsilon_{t+1}\right) = J\left(X_t, X_{t+1}, \varepsilon_{t+1}\right) / M\left(X_t, X_{t+1}, \varepsilon_{t+1}\right)$ so that we can apply the method properly.

Let us denote$\hat{X}_t = X_t - \bar{X}$ as the deviation of the vector of state variables from its non-stochastic steady-state value. It will also be useful to define:

$$m\left(X_t, X_{t+1}, \varepsilon_{t+1}\right) = \log\left[M\left(X_t, X_{t+1}, \varepsilon_{t+1}\right)\right]$$

$$f\left(X_t, X_{t+1}, \varepsilon_{t+1}\right) = \log\left[F\left(X_t, X_{t+1}, \varepsilon_{t+1}\right)\right]$$

The author then approximates these functions $D\left(X_t, X_{t+1}, \varepsilon_{t+1}\right)$,

$m(X_t, X_{t+1}, \varepsilon_{t+1})$ and $f(X_t, X_{t+1}, \varepsilon_{t+1})$ around the non-stochastic steady-state:

$$D(X_t, X_{t+1}, \varepsilon_{t+1}) \approx D_0 + D_X \hat{X}_t + D_{X'} \hat{X}_{t+1} + D_\varepsilon \varepsilon_{t+1} \tag{3-1}$$

$$m(X_t, X_{t+1}, \varepsilon_{t+1}) \approx m_0 + m_X \hat{X}_t + m_{X'} \hat{X}_{t+1} + m_\varepsilon \varepsilon_{t+1} \tag{3-2}$$

$$f(X_t, X_{t+1}, \varepsilon_{t+1}) \approx f_0 + f_X \hat{X}_t + f_{X'} \hat{X}_{t+1} + f_\varepsilon \varepsilon_{t+1} \tag{3-3}$$

where the coefficient matrices are these partial derivatives evaluated at the steady-state. These derivatives are a function of the parameters of the model and non-stochastic steady-state values of the variables. Note that by the way the system was written we have that $D_0 = 0_{N_D \times 1}$ and $m_0 + f_0 = 0$. This way, we can rewrite the approximated equilibrium conditions as:

$$D_X \hat{X}_t + D_{X'} \hat{X}_{t+1} + D_\varepsilon \varepsilon_{t+1} = 0_{N_D \times 1} \tag{3-4}$$

$$E_t \left[\exp\left\{ m_X \hat{X}_t + m_{X'} \hat{X}_{t+1} + m_\varepsilon \varepsilon_{t+1} \right\} \times \exp\left\{ f_X \hat{X}_t + f_{X'} \hat{X}_{t+1} + f_\varepsilon \varepsilon_{t+1} \right\} \right] = 1_{N_F \times 1}$$

Applying the logarithm on the second set of equations:

$$\log E_t \left[\exp\left\{ (m_X + f_X) \hat{X}_t + (m_{X'} + f_{X'}) \hat{X}_{t+1} + (m_\varepsilon + f_\varepsilon) \varepsilon_{t+1} \right\} \right] = 0_{N_F \times 1} \tag{3-5}$$

Now we guess that the approximated solution of the model takes the form:

$$\hat{X}_{t+1} = H_0 + H_X \hat{X}_t + H_\varepsilon \varepsilon_{t+1}$$

Substituting the guess in the equilibrium conditions (3-4) and (3-5), we get:

$$D_X \hat{X}_t + D_{X'} \left(H_0 + H_X \hat{X}_t + H_\varepsilon \varepsilon_{t+1} \right) + D_\varepsilon \varepsilon_{t+1} = 0_{N_D \times 1}$$

$$\log E_t \left[\exp\left\{ (m_X + f_X) \hat{X}_t + (m_{X'} + f_{X'}) \left(H_0 + H_X \hat{X}_t + H_\varepsilon \varepsilon_{t+1} \right) + (m_\varepsilon + f_\varepsilon) \varepsilon_{t+1} \right\} \right] = 0_{N_F \times 1}$$

Under the hypothesis that the variables of vector $\hat{X}_t$ and the shocks in ε_t are jointly normally distributed, we can write:

$$D_X \hat{X}_t + D_{X'} \hat{X}_{t+1} + D_\varepsilon \varepsilon_{t+1} = 0_{N_D \times 1} \tag{3-6}$$

$$(m_X + f_X)_i \hat{X}_t + (m_{X'} + f_{X'})_i E_t \left(\hat{X}_{t+1} \right) + \frac{1}{2} \Gamma^\sigma_{ii} = 0 \qquad for\ i = 1, \ldots, N_F \tag{3-7}$$

where Γ^σ_{ii} is the i-th element of the diagonal of matrix $\Gamma^\sigma = (m_\varepsilon + f_\varepsilon + (m_{X'} + f_{X'}) H_\varepsilon)(m_\varepsilon + f_\varepsilon + (m_{X'} + f_{X'}) H_\varepsilon)'$ and $(m_X + f_X)_i$ is the i-th line of matrix $(m_X + f_X)$. The system of equations (3-6)-(3-7) is linear in the vector of variables and can be solved with well known methods, like Sims' (2001) gensys algorithm. The problem is that matrix H_ε is not known, as it is part of the solution of the model. This means that the equations of the system depend on the values of their own solutions. This way, the author suggests an algorithm that iteratively tries to find a "fixed point". The solution algorithm is given by:

- **First Step**: Approximate functions $D\left(X_t, X_{t+1}, \varepsilon_{t+1}\right)$, $m\left(X_t, X_{t+1}, \varepsilon_{t+1}\right)$ and $f\left(X_t, X_{t+1}, \varepsilon_{t+1}\right)$ around the non-stochastic steady-state to find the coefficients matrices of equations (3-1), (3-2) and (3-3).
- **Second Step**: For the first iteration $j = 0$, use the gensys algorithm to solve the system (3-6)-(3-7) ignoring the term that depends on H_ε:

  $$D_X \hat{X}_t + D_{X'} \hat{X}_{t+1} + D_\varepsilon \varepsilon_{t+1} = 0_{N_D \times 1}$$

  $$\left(m_X + f_X\right) \hat{X}_t + \left(m_{X'} + f_{X'}\right) E_t\left(\hat{X}_{t+1}\right) = 0_{N_F \times 1}$$

  This will deliver a first solution for $H_0^{(0)}$, $H_X^{(0)}$ and $H_\varepsilon^{(0)}$. Note that these are the solutions for the traditional first-order approximations.
- **Third Step**: For iterations $j > 0$, solve the system

  $$D_X \hat{X}_t + D_{X'} \hat{X}_{t+1} + D_\varepsilon \varepsilon_{t+1} = 0_{N_D \times 1}$$

  $$\left(m_X + f_X\right)_i \hat{X}_t + \left(m_{X'} + f_{X'}\right)_i E_t\left(\hat{X}_{t+1}\right) + \frac{1}{2}\Gamma_{ii}^{\sigma(j)} = 0 \qquad for\ i = 1, \ldots, N_F$$

  where $\Gamma^{\sigma(j)} = \left(m_\varepsilon + f_\varepsilon + \left(m_{X'} + f_{X'}\right) H_\varepsilon^{(j-1)}\right)\left(m_\varepsilon + f_\varepsilon + \left(m_{X'} + f_{X'}\right) H_\varepsilon^{(j-1)}\right)'$, until all solution matrices $H_0^{(j)}$, $H_X^{(j)}$ and $H_\varepsilon^{(j)}$ have converged.

If there are no state variables that affect second or higher order moments, like stochastic volatility or time varying risk-aversion, this method will converge in exactly two iterations because the only difference on the second iteration would be an additional vector of constants on the on the second set of equations, which would only change the solution of H_0 compared with the one from the first iteration, but this is not the case for all functional forms, as we will see later when working with Epstein-Zin preferences.

Even though $\hat{X}_t$ represents the deviation from the non-stochastic steady-state of the variables, its steady-state value will not be zero. The risk adjustment changes the level of the variables in order to approximate around the stochastic steady-state. If X_t contains the log of the variables, then the steady-state of $\hat{X}_t$ can be interpreted as the percentage deviation of the stochastic steady-state from the non-stochastic one.

The pricing of zero-coupon bonds is given by:

$$b_{n,t} = E_t\left[\exp\left(m_0 + m_X \hat{X}_t + m_{X'} \hat{X}_{t+1} + m_\varepsilon \varepsilon_{t+1}\right) \times b_{n-1,t+1}\right]$$

Substituting the approximation of the SDF, guessing that the log of bond prices are an affine function of the state variables and using log-normality we can match coefficients to find:

$$A_n = A_{n-1}+m_0+(m_{X'} + B_{n-1})\, H_0+\frac{1}{2}\left[m_\varepsilon + (m_{X'} + B_{n-1})\, H_\varepsilon\right]\left[m_\varepsilon + (m_{X'} + B_{n-1})\, H_\varepsilon\right]'$$

$$B_n = m_X + (m_{X'} + B_{n-1})\, H_X$$

where each yield to maturity is computed as:

$$r_{n,t} = -\frac{1}{n}\left(A_n + B_n\hat{X}_t\right)$$

3.4
Calibration and Comparison of Methods

For a baseline calibration, we use the estimated parameters values from Castro *et al* (2011). We calibrate the inverse elasticity of labor supply $\varphi = 1$, the elasticity of substitution between differentiated goods to be $\varepsilon = 11$, which generates a 10% steady state price markup in the differentiated goods market, and set the monetary policy shock persistence to be $\rho_v = 0$. Inverse elasticity of substitution is $\sigma = 1.3$ and $\rho_\pi = 2.43$. For θ we need to adjust their estimate to a different time frequency. They have a model with quarterly frequency, so that their value of $\theta = 0.74$ implies that firms reset their prices, on average every, 3.84 quarters or 11.53 months. Since our model has a monthly frequency, based on this value, our calibration is $\theta = 0.9133$, also implying an average price duration of 11.53 months. The frequency adjustment was also applied to the interest rate smoothing parameter, $\rho_i = 0.9244$, and to the persistence of technology shocks, $\rho_a = 0.9655$. The remaining parameters were set to $\beta = 0.9895$, $\sigma_a = 0.009$ and $\sigma_v = 0.004$ to match the empirical level and slope of the Brazilian yield curve over the full sample period with the steady states values of the yield curve of the model. Table 3.1 summarizes the calibrated parameters.

Table 3.1: Calibrated Parameters

Parameter	β	σ	φ	ε	θ	ρ_π	ρ_i	ρ_a	σ_a	ρ_v	σ_v
Baseline	0.9895	1.3	1	11	0.9133	2.43	0.9244	0.9655	0.009	0	0.004

To evaluate the dynamics of the responses, figure 3.1 shows the impulse response function for several variables for several methods. First- and second-

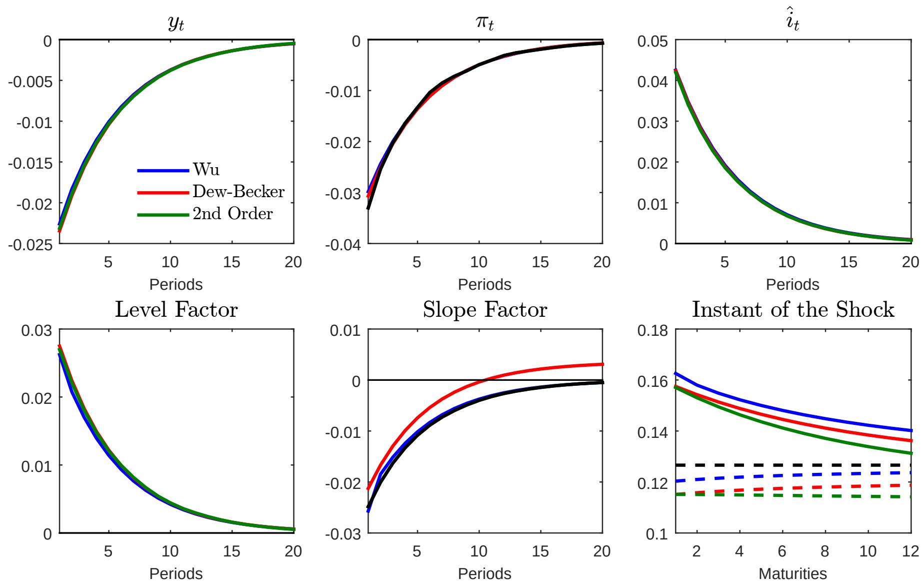

Figure 3.1: Monetary Policy Shock - Wu's method in blue, Dew-Becker's method in red, first-order method in black and second-order method in green.

Table 3.2: Steady-state values for different solution methods

Variable	Solution Method			
	First-order	Wu	Dew-Becker	Second-order
$\hat{y}_t$	0	0	-1.37%	-1.57%
$\hat{\pi}_t$	0	0	-0.47%	-0.47%
$\hat{i}_t$	0	0	-1.15%	-1.15%
$r_{1,t}$	12.72%	12.15%	11.51%	11.51%
$r_{6,t}$	12.72%	12.35%	11.75%	11.47%
$r_{12,t}$	12.72%	12.45%	11.87%	11.41%

order approximations were computed in dynare[2], which uses the method from Schmitt-Grohé and Uribe (2004). We can see that all the methods generate an almost identical impulse response function, but the level of the variables are different. This can be seen in table 3.2, which shows the steady state value for all methods. We can see that in the first order method all yields are the same due to the risk-neutral behavior generated by the approximation. Wu's method only adds a constant on the yield equations, so they are the only ones that have their values changed while the macro variables have no precautionary saving effect. Dew-Becker's method does a great job in approximating the steady state generated by the second order approximation. In his paper, the author computes the the Euler equation approximation errors for a simple RBC model and shows that they are very competitive with the ones from a second-order approximation. Malkhozov and Shamloo (2010) also point out

[2]The impulse response functions for the second-order approximation are explosive for this calibration, so they were computed using the prunning method.

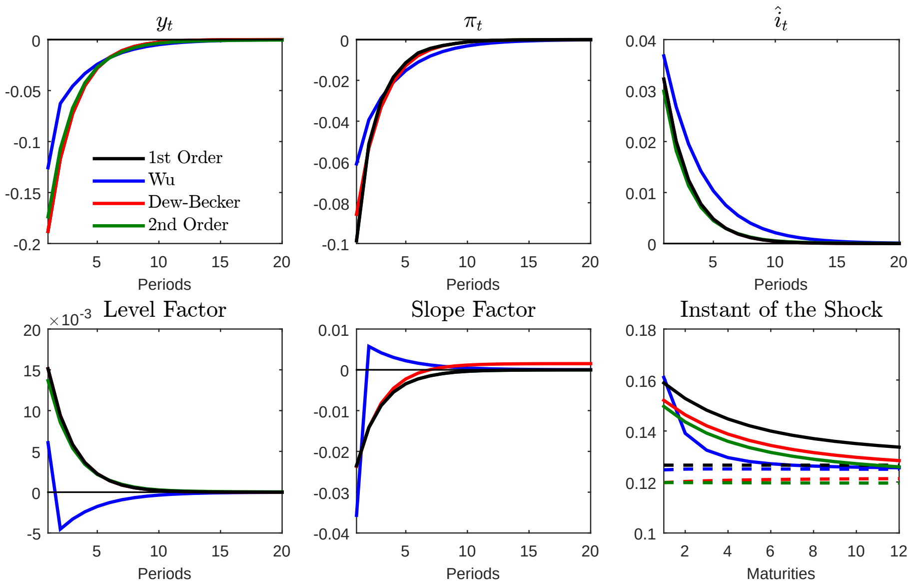

Figure 3.2: Monetary Policy Shock in the alternative calibration with $\sigma = 0.1$ and $\varphi = 1.5$ - Wu's method in blue, Dew-Becker's method in red, first-order method in black and second-order method in green.

that, numerically, the coefficients solutions from the risk-adjusted method and second-order approximations are very similar while the coefficients associated to quadratic terms of the second-order approximation are close to zero.

Although the solutions have a diffent shape, specially for the yield curve and its steady state, the response of the macro variables are very close to each other for all the solution methods. This might be a result of the calibration chosen, which may not generate enough curvature in the system in order for the solutions to be significantly different. Figure 3.2 shows the same analysis for a different calibration, where we lower the intertemporal elasticity of substitution to $\sigma = 0.1$ and increased the Frisch elasticity of labor to $\varphi = 1.5$. It is possible to see that Wu's method now has a slightly diffent trajectory for the solutions of macro variables and for the yield curve, specially in the first instant of the shock where longer maturities have a smaller response when compared to the other solution methods.

3.5
Comparing Calibrations

Now we go back to our baseline calibration and compute moments and impulse response functions for another set of alternative calibrations in order to analyze the effect of different parameter values over the system dynamics and steady state values. We are now specially interested in the effect of changes in the persistence of monetary policy shocks ρ_v, the smoothing component

ρ_i of the interest rate rule. We are going to use Wu's method to generate the IRFs for alternative calibrations in order to understand how they changes with different values of interest rate persistence. Although this solution method is not appropriate for the level of the variables model, it does well with deviations from the steady state. We set ρ_v to be either 0 or 0.7 and ρ_i to be either 0 or 0.9244[3].

Simulating the model with the baseline calibration we could see that it does not replicate the moments of the data very well but some stylized facts, although not in the same magnitude, appear. The volatility of the shorter ends of the curve are higher than the longer ends and they are all highly correlated with the central bank's policy rate. We can also see that a higher value of ρ_v is necessary to generate the high autocorrelation of yields found in the data and it helps increase the volatility of the longer ends of the curve, although it generates a volatility for the macro variables that are a lot bigger than the ones found in the data, but this is probably due to small number of shocks.

The first and second line of figure 3.3 show the impulse response functions of the level and the slope of the yield curve, respectively, after a monetary policy shock for a few different parameters values. Revisiting our empirical VAR exercise, we can see that the model can capture the shape of the IRF's[4]. For the first sample period, the level factor has a exponentially decreasing IRF and there is a negative impact on the slope. For the model to capture this movement there must be a source of persistence from either the shock itself or from a smoothing component of the interest rate rule, but not from both. In the case where the two sources of persistence are present, we have a response with a shape that resembles the one in the second sample period of the VAR study, where the level factor has a hump-shaped response and the slope factor has an initial positive impact on the slope, but it rapidly goes negative. This means that, due to the persistence of the shock, there is a stronger initial impact on the long end of the curve, making the slope increase, but since this effect is less persistent on the long ends of the curve, the longer yields return faster to their steady state, making the curve flatter.

The third line of figure 3.3 shows the curve in its steady state (dashed lines) and on the instant of the impact of the monetary shock (solid lines).

[3]We also did this for different values of ρ_π and σ_v to see how they may change the shape of the IRFs. Even large changes in the value of ρ_π had small effects on the shape of the IRFs, but it is worth noting that a higher value of ρ_π generates smaller steady-state yields. Changes in σ_v have no effect on the shape of the IRFs, only on their scale, and a higher value of this parameter also generates smaller steady-state yields.

[4]We can not compare the level of the IRFs since the variables are not in the same unit of measure. The rates on the model are in annualized percentages and the VAR model has a factor extracted from principal component analysis.

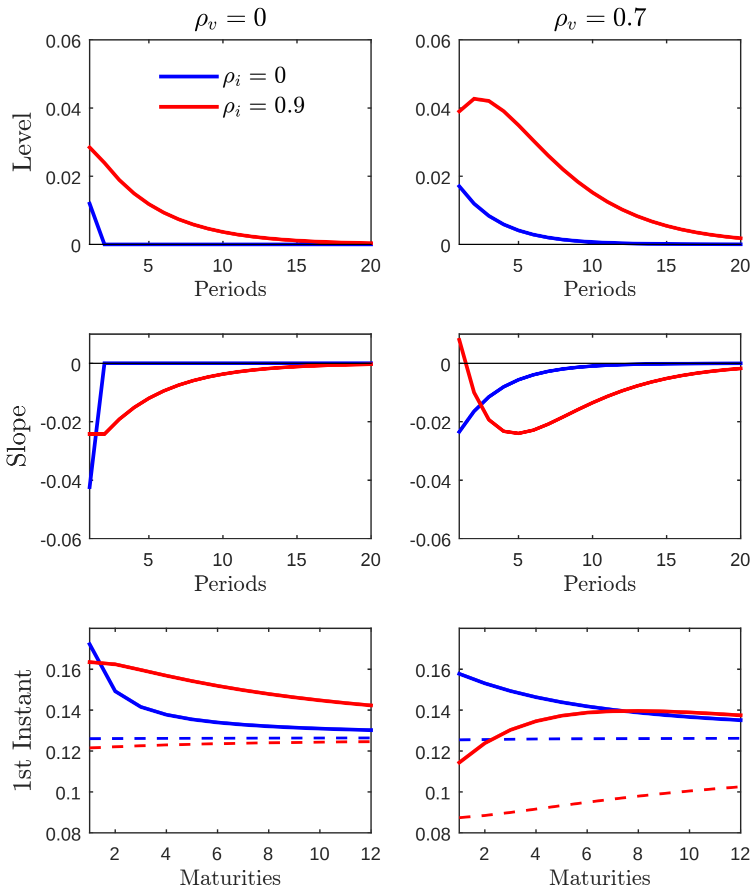

Figure 3.3: Impulse responses for different calibrations

We can see that that a higher persistence, either from the shock or from the policy rule, decreases the impact of the shock on the short end of the curve and increases it on the longer ends. The impact on the steady state of the curve is very small and it is due to indirect changes in the solution coefficients of the macro variables since the parameters ρ_v and ρ_i have no direct effect on the constants that define the steady state. In the presence of both sources of persistence this effect changes to the one described previously, where the initial effect over the slope is positive, but notice that in the instant of the shock the curve now has a concave shape as opposed to the convex one when there is only one or no source of persistence.

Table 3.3: Comparison of Unconditional Moments

Data (2004m01-2015m01)				
x_t	σ_{x_t}	$\frac{\sigma_{x_t}}{\sigma_{\hat{i}_t}}$	$\rho_{x_t,x_{t-1}}$	$\rho_{x_t,\hat{i}_t}$
$r_{1,t}$	0.0337	1.003	0.9891	0.9986
$r_{6,t}$	0.0329	0.979	0.9850	0.9754
$r_{12,t}$	0.0313	0.9315	0.9783	0.9445
$\hat{i}_t$	0.0336	1	0.9870	1
$\hat{y}_t$	0.0340	1.011	0.8372	0.0014
$\hat{\pi}_t$	0.0286	0.851	0.5708	-0.0090

Model - Baseline Calibration				
x_t	σ_{x_t}	$\frac{\sigma_{x_t}}{\sigma_{\hat{i}_t}}$	$\rho_{x_t,x_{t-1}}$	$\rho_{x_t,\hat{i}_t}$
$r_{1,t}$	0.0744	1	0.8238	1
$r_{6,t}$	0.0492	0.6613	0.8331	0.9936
$r_{12,t}$	0.0336	0.4516	0.8482	0.9692
$\hat{i}_t$	0.0744	1	0.8238	1
$\hat{y}_t$	0.0484	0.6505	0.8648	-0.9311
$\hat{\pi}_t$	0.0540	0.7258	0.8179	-0.9359

Model - $\rho_v = 0.7$				
x_t	σ_{x_t}	$\frac{\sigma_{x_t}}{\sigma_{\hat{i}_t}}$	$\rho_{x_t,x_{t-1}}$	$\rho_{x_t,\hat{i}_t}$
$r_{1,t}$	0.1308	1	0.9653	1
$r_{6,t}$	0.1116	0.853	0.9204	0.9744
$r_{12,t}$	0.0816	0.624	0.9011	0.9553
$\hat{i}_t$	0.1308	1	0.9653	1
$\hat{y}_t$	0.1366	1.044	0.8663	-0.9200
$\hat{\pi}_t$	0.2016	1.541	0.8415	-0.8878

4
Empirics

Now that we understand how the essentially affine solution method works, the objective of this section is to estimate a model using this method and investigate what might have been the reason for the change in the way that the Brazilian yield curve responds to monetary policy shocks. First, we make a few changes to model presented in the previous section in order to get a better fit of the data. We are going to include Epstein-Zin preferences with time-varying risk aversion, habit formation and price indexation.

4.1
Medium-Scale New Keynesian Model

4.1.1
Household

The household has Epstein-Zin (1989) preferences, which are recursive between consumption C_t, labor N_t and the certainty equivalent of future utility $R_t\left(V_{t+1}\right)$.

$$V_t = \left\{(1-\beta)\, U\left(C_t, \bar{C}_{t-1}, N_t, Z_t\right) + \beta R_t\left(V_{t+1}\right)^{1-\sigma}\right\}^{\frac{1}{1-\sigma}}$$

$$R_t\left(V_{t+1}\right) = E_t\left(V_{t+1}^{1-\gamma_t}\right)^{\frac{1}{1-\gamma_t}}$$

Recursive preferences allow to separate parameters that define the intertemporal elasticity of substitution and relative risk aversion which is important in the asset pricing literature. $\bar{C}_{t-1}$ is the aggregate consumption from last period, which the household takes as given, σ is the inverse of the intertemporal elasticity of substitution and γ_t is the coefficient of relative risk aversion, which is allowed to be vary over time following and AR(1) process:

$$\gamma_t = \left(1-\rho_\gamma\right)\bar{\gamma} + \rho_\gamma \gamma_{t-1} + \sigma_\gamma \varepsilon_t^\gamma$$

Even though shocks to γ_t might have small effects on consumption and labor, as we will see later, their volatility will still be priced as a risk and taken into account when adjusting for the precautionary savings effect. The instantaneous utility function has a multiplicative habit formation term to ensure that changes in risk aversion are exclusive due to shocks to γ_t. The function is given by:

$$U\left(C_t, \bar{C}_{t-1}, N_t, Z_t\right) = \frac{\left(C_t^{\eta}\bar{C}_{t-1}^{1-\eta}\right)^{1-\sigma}}{1-\sigma} + \varphi\frac{\left(C_t^{H}\right)^{1-\sigma}}{1-\sigma}$$

The household has CRRA preferences over "market-work-time" goods C_t and "non-market-work-time" goods C_t^H. The production function of the latter is $C_t^H = Z_t\left(N_t^H\right)^{\alpha_H}$. The household has a labor endowment of $\bar{H}$ which can be allocated either for production of market goods, N_t, or non-market goods, N_t^H. We can rewrite the the instantaneous utility function as:

$$U\left(C_t, \bar{C}_{t-1}, N_t, Z_t\right) = \frac{\left(C_t^{\eta}\bar{C}_{t-1}^{1-\eta}\right)^{1-\sigma}}{1-\sigma} + \varphi Z_t^{1-\sigma}\frac{\left(\bar{H}-N_t\right)^{\alpha_H(1-\sigma)}}{1-\sigma}$$

The reason that the productivity enters the utility function is to assure that the marginal product of labor in both sectors stays proportional to Z_t. We know that the household optimization problem yields a consumption Euler equation, which is also interpreted as the stochastic discount factor, and a labor supply curve. The intratemporal condition of optimality is:

$$\frac{W_t^n}{P_t} = -\frac{{\partial V_t}/{\partial N_t}}{{\partial V_t}/{\partial C_t}} = \frac{\varphi\alpha_H Z_t^{1-\sigma}\left(\bar{H}-N_t\right)^{\alpha_H(1-\sigma)-1}}{\eta C_t^{\eta(1-\sigma)-1}\bar{C}_{t-1}^{(1-\sigma)(1-\eta)}}$$

From this point on we denote $W_t = \frac{W_t^n}{P_t}$. The nominal stochastic discount factor is given by:

$$M_{t,t+1} = \frac{{\partial V_t}/{\partial C_{t+1}}}{{\partial V_t}/{\partial C_t}}\frac{P_t}{P_{t+1}}$$

To write the stochastic discount factor for Epstein-Zin preferences recursively we need to define two auxiliary variables. The first one is $J_t = \frac{V_t}{{\partial V_t}/{\partial C_t}}$, which works as a measure of wealth, measuring utility in terms of consumption goods. The second one is the "cum-dividend" return on this wealth measure, defined as:

$$R_{J,t+1} = \frac{J_{t+1}}{J_t - U_t U_{c,t}^{-1}}$$

Where $U_{c,t}$ is the marginal utility of consumption. With some manipulation, we can use these auxiliary variables to write:

$$M_{t,t+1} = \beta^{\frac{1-\gamma_t}{1-\sigma}}\left[\left(\frac{C_{t+1}}{C_t}\right)^{\eta(1-\sigma)-1}\left(\frac{C_t}{C_{t-1}}\right)^{(1-\sigma)(1-\eta)}\right]^{\frac{1-\gamma_t}{1-\sigma}} R_{J,t+1}^{\frac{\sigma-\gamma_t}{1-\sigma}}\Pi_{t+1}^{-1}$$

$$J_t = U_t U_{c,t}^{-1} + E_t\left[\frac{{\partial V_t}/{\partial C_{t+1}}}{{\partial V_t}/{\partial C_t}}J_{t+1}\right]$$

Note that when $\eta = 1$ (no habit formation) and $\gamma_t = \sigma$ (relative risk aversion is constant and equal to IES) we get the stochastic discount factor for the usual time separable CRRA preferences.

4.1.2 Final Good Producers

The final producer operates in perfect competition and the inputs are bought in a monopolistically competitive market. The production function of the final good producer is given by:

$$Y_t = \left(\int_0^1 Y_t\left(i\right)^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}}$$

where ε is the elasticity of technical substitution between different inputs. The final goods producer takes the input prices $P_t\left(i\right)$ and his selling price P_t as given. The producer's problem is to choose how much of each input to buy in order to maximize profits. The solution to this problem yields a demand for intermediate goods and a price index, which are respectively given by:

$$Y_t\left(i\right) = \left(\frac{P_t\left(i\right)}{P_t} \right)^{-\varepsilon} Y_t$$

$$P_t = \left(\int_0^1 P_t\left(i\right)^{1-\varepsilon} di \right)^{\frac{1}{1-\varepsilon}}$$

4.1.3 Intermediate Good Producers

There is a continuum of intermediate producers, indexed by $i \in [0,1]$, operating in a monopolistically competitive market. All of them have a production function of the form:

$$Y_t\left(i\right) = Z_t N_t\left(i\right)$$

where $N_t\left(i\right)$ is the amount of labor employed by the i-th firm and Z_t is the level of productivity, common to all intermediate firms. These producers operate under a Calvo (1983) pricing scheme, in which a fraction $1-\theta$ of intermediate firms are allowed to reset their prices at each period and the remaining fraction θ have their prices indexed by last period's inflation, following the rule:

$$P_{t+k|t}\left(i\right) = \begin{cases} P_t^{\star} \left(\prod_{j=1}^{k} \Pi_{t+j-1}^{\tau} \right) & , \ k \geq 1 \\ P_t^{\star} & , \ k = 0 \end{cases}$$

Intermediate producers face two problems. The first one is choosing the amount of labor to hire in order to minimize costs, given his demand (sourcing problem). The second one is setting his price in order to maximize his stream of expected profits (pricing problem). The solution to the sourcing problem gives us the optimal real marginal cost:

$$MC_t = \frac{W_t}{Z_t}$$

The first order condition of the pricing problem can be written recursively as:

$$\frac{P_t^\star}{P_t} F_{1,t} = \frac{\varepsilon}{\varepsilon - 1} F_{2,t}$$

$$F_{1,t} = Y_t + \theta \Pi_t^{\tau(1-\varepsilon)} E_t \left\{ M_{t,t+1} \Pi_{t+1}^{\varepsilon} F_{1,t+1} \right\}$$

$$F_{2,t} = Y_t MC_t + \theta \Pi_t^{-\tau\varepsilon} E_t \left\{ M_{t,t+1} \Pi_{t+1}^{1+\varepsilon} F_{2,t+1} \right\}$$

Note that if $\theta = 0$ (perfectly flexible prices) we have:

$$P_t^\star = \frac{\varepsilon}{\varepsilon - 1} P_t MC_t$$

4.1.4
Central Bank

To help the model fit the data, we allow the central bank to react to deviations of inflation from its target, to deviations of the level of output from its steady-state and to the gross rate of output growth. Hordhal, Tristani e Vestin (2006) point out that a strong interest rate smoothing component is very important to help match yield curve data. So the central bank follows a interest rate rule given by:

$$R_t = R_{t-1}^{\rho_R} \left[\bar{R} \left(\frac{\Pi_t}{\Pi_t^\star} \right)^{\phi_\pi} \left(\frac{Y_t}{\bar{Y}} \right)^{\phi_Y} \left(\frac{Y_t}{Y_{t-12}} \right)^{\phi_{dY}} \right]^{1-\rho_R} \exp\left(\varepsilon_t^R \right)$$

Where $\bar{Y}$ is the steady state value of output and $\Pi_t^\star$ is the time-varying gross inflation target, which follows an AR(1) process given by:

$$\ln\left(\Pi_t^\star\right) = \ln\left(\bar{\Pi}^\star\right) + \rho_\pi \ln\left(\Pi_{t-1}^\star\right) + \sigma_\pi \varepsilon_t^\pi$$

4.1.5

Aggregation and Market-Clearing

The calvo pricing scheme implies that we can write the price index as:

$$1 = (1-\theta)\left(\frac{P_t^\star}{P_t}\right)^{1-\varepsilon} + \theta\left(\frac{\Pi_{t-1}^{\tau}}{\Pi_t}\right)^{1-\varepsilon}$$

Market clearing in the labor market implies that

$$Y_t = \frac{Z_t}{S_t} N_t$$

Where $S_t = \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon} di$ is a term associated with the inefficiency generated by price dispersion. The dynamics of this term can be found using a result from the Calvo pricing scheme:

$$S_t = (1-\theta)\left(\frac{P_t^\star}{P_t}\right)^{-\varepsilon} + \Pi_{t-1}^{-\varepsilon\tau}\Pi_t^{\varepsilon}\theta S_{t-1}$$

The resource constraint of the economy is:

$$Y_t = C_t$$

4.2 Essentially Affine Method for Epstein-Zin Preferences

Dew-Becker (2012) points out that we can take advantage of the functional form of Epstein-Zin preferences together with the assumption of log-normality in order to capture higher order effects, even in linear form. The log of the stochastic discount factor is given by:

$$m\left(X_{t+1}, X_t, \varepsilon_{t+1}\right) = \zeta_t \ln(\beta) + \zeta_t\left[u_{c,t+1} - u_{c,t}\right] + (\zeta_t - 1)\, r_{j,t+1} - \pi_{t+1}$$

where we defined $\zeta_t = \frac{1-\gamma_t}{1-\sigma}$ to save on notation. We must solve:

$$E_t\left[\exp(m_{t+1}) \times F\left(X_{t+1}, X_t, \varepsilon_{t+1}\right)\right] = 1$$

In particular, one of the equation in $F\left(X_{t+1}, X_t, \varepsilon_{t+1}\right)$ is the Euler equation for the return on wealth:

$$1 = E_t\left[M_{t,t+1}\pi_{t+1}R_{J,t+1}\right]$$

Taking logs on both sides, using a selection matrix $\Gamma_x X_t = x_t$, and using log-normality of the variables in the system, we get:

$$0 = \zeta_t \left(\Gamma_{u_c} + \Gamma_{r_j}\right) E_t \left(\hat{X}_{t+1}\right) - \zeta_t \Gamma_{u_c} \hat{X}_t + \frac{1}{2} Var_t \left\{\zeta_t \left(\Gamma_{u_c} + \Gamma_{r_j}\right) \hat{X}_{t+1}\right\}$$

Guessing that the system has a solution on the form of:

$$\hat{X}_{t+1} = H_0 + H_X \hat{X}_t + H_\varepsilon \varepsilon_{t+1}$$

we can substitute to get:

$$0 = \zeta_t \left(\Gamma_{u_c} + \Gamma_{r_j}\right) E_t \left(\hat{X}_{t+1}\right) - \zeta_t \Gamma_{u_c} \hat{X}_t + \frac{1}{2} \zeta_t^2 \left(\Gamma_{u_c} + \Gamma_{r_j}\right) H_\varepsilon H_\varepsilon' \left(\Gamma_{u_c} + \Gamma_{r_j}\right)' \tag{4-1}$$

Now notice that all terms are multiplied by ζ_t, so we can simplify to get:

$$0 = \left(\Gamma_{u_c} + \Gamma_{r_j}\right) E_t \left(\hat{X}_{t+1}\right) - \Gamma_{u_c} \hat{X}_t + \frac{1}{2} \left(\Gamma_{u_c} + \Gamma_{r_j}\right) H_\varepsilon H_\varepsilon' \left(\Gamma_{u_c} + \Gamma_{r_j}\right)' \zeta_t$$

So the Euler equation for the return on wealth is linear in the state variables. Now we can do the same to the other Euler equations in $F\left(X_{t+1}, X_t, \varepsilon_{t+1}\right)$, in a general form, to get:

$$\begin{aligned} 0 &= \left[\zeta_t \left(\Gamma_{u_c} + \Gamma_{r_j}\right) - \Gamma_{r_j} - \Gamma_\pi + f_{X'}\right] E_t \left(\hat{X}_{t+1}\right) + \left(f_X - \zeta_t \Gamma_{u_c}\right) \hat{X}_t \\ &+ \frac{1}{2} \left(f_{X'} - \Gamma_{r_j} - \Gamma_\pi\right) H_\varepsilon H_\varepsilon' \left(f_{X'} - \Gamma_{r_j} - \Gamma_\pi\right)' \\ &+ \zeta_t \left[\left(\Gamma_{u_c} + \Gamma_{r_j}\right) H_\varepsilon H_\varepsilon' \left(f_{X'} - \Gamma_{r_j} - \Gamma_\pi\right)'\right] \\ &+ \frac{1}{2} \zeta_t^2 \left(\Gamma_{u_c} + \Gamma_{r_j}\right) H_\varepsilon H_\varepsilon' \left(\Gamma_{u_c} + \Gamma_{r_j}\right)' \end{aligned}$$

Here we can see that the risk-aversion coefficient has non-linear precautionary savings effect on the system, but if we rearrange equation (4-1) substitute it back in this last expression we get:

$$\begin{aligned} 0 &= \left(f_{X'} - \Gamma_{r_j} - \Gamma_\pi\right) E_t \left(\hat{X}_{t+1}\right) \\ &+ \left\{f_X + \left[\left(\Gamma_{u_c} + \Gamma_{r_j}\right) H_\varepsilon H_\varepsilon' \left(f_{X'} - \Gamma_{r_j} - \Gamma_\pi\right)'\right] \Gamma_\zeta\right\} \hat{X}_t \\ &+ \frac{1}{2} \left(f_{X'} - \Gamma_{r_j} - \Gamma_\pi\right) H_\varepsilon H_\varepsilon' \left(f_{X'} - \Gamma_{r_j} - \Gamma_\pi\right)' + \left[\left(\Gamma_{u_c} + \Gamma_{r_j}\right) H_\varepsilon H_\varepsilon' \left(f_{X'} - \Gamma_{r_j} - \Gamma_\pi\right)'\right] \bar{\zeta} \end{aligned}$$

Which is linear in the state variables. So we have the N_D equations from the non-expectational block, $N_F - 1$ equations from the forward looking block, that will use the form of this last expression, and the Euler equation for the return on wealth as equilibrium conditions for model to be estimated.

4.3 Estimation

The data is the same as the one described in section 2, but the yield curve data is observable as individual yields and not summarized with principal components. Again, we separate our sample between two time periods, the first one from January 2004 until December 2010 and the second on from January 2011 until January 2015, due to a change in the political regime.

The model is estimated using standard bayesian methods. Since our model has a linear solution, we can evaluate its likelihood function using the Kalman filter. The state equation is the solution of the model dynamics and the observation equation is a selection matrix for the observable variables and the affine bond pricing equations. As is commonly done in the literature and to avoid stochastic singularity, we assume that yields are observed with a measurement error, all independent of each other.

The posterior mode is found using numerical optimization and from that point we run the Random Walk Metropolis algorithm (RWM), as described by An and Schoferheide (2007), to approximate the posterior distribution. We used a normally distributed transition kernel with zero mean and for the covariance matrix of the we used the inverse of the hessian of the posterior evaluated at the mode of the posterior distribution, but scaled in order to achieve around a 23% rejection rate. At each iteration, the solution method did not take more than 6 steps to converge and if in any of them resulted in undetermined or non-existing solution of the model, we set its likelihood to minus infinity, although that did not occur when starting the chain at the posterior mode. For each sample period, finding the posterior mode took around 20 minutes and running two million iterations of the RWM took around five days of computing. To analyze the sample from the posterior, we eliminate the first quarter of each chain leaving us with 1.5 million sampled posterior observations for each sample period.

We calibrated four parameters that are difficult to estimate based on Castro et al (2011), but adjusting to a monthly time frequency. When trying to find the posterior mode without calibrating them we found values for them that are very far away from the ones found in the literature. We set the monthly intertemporal time preference to $\beta = 0.9963$, which represents a 4.55% annualized rate, the time endowment $\bar{H} = 720$, the fraction of firms that do not reset their prices at each period to $\theta = 0.9045$ and the technical elasticity of substitution between inputs for the final goods producer to $\varepsilon = 11$, which generates a price markup of 10% for the intermediate producers. All remaining parameters, including the variance of measurement errors, are estimated.

The priors chosen were influenced by Castro et al (2011) not only on their distribution but also on their means and variances. We set prior means to be close to the estimated values of their model but keeping a relatively large variance in order to lower the importance of the prior distribution on the posterior. For parameters that are not in their model, we chose a distribution depending on the support of the parameter space and the mean and variance based on the literature. For example, other papers that estimate parameters of an Epstein-Zin utility function usually find values for the risk-aversion coefficient that are much larger than the elasticity of substitution, so the prior mean for $\bar{\gamma}$ is larger than the one from σ. Table 4.1 shows the prior and posterior distributions moments and figures 4.1 and 4.2 plot the prior densities and the sampled posteriors for all estimated parameters for both sample periods.

Although some of the posterior modes are reasonably close to their prior means, this was expected since prior means were chosen based on estimates from other papers. Also, posterior standard deviations are considerably smaller than prior standard deviations. One exception is the posterior of the volatility of risk aversion shocks which has a very small standard deviation on the first sample and a vey large one on the second sample, but this will be assessed soon.

The estimates of both periods for the intertemporal elasticity of substitution σ has overlapping credibility intervals, so we can not say that its value has changed from one period to another. Although the estimated values from other studies that also estimate DSGE models for Brazil are smaller, they use different functional forms. The steady-state value of the risk-aversion coefficient $\bar{\gamma}$ for the second sample period is larger than the first one. Intuitively, this might mean that consumers have a higher perception of risk over the second-sample period. Their estimates are much larger than the intertemporal elasticity of substitution. This is a common result in models that work with Epstein-Zin preferences, such as Binsbergen (2012). The persistence of the risk-aversion coefficient is very high on both sample periods and although the mode of the volatility of their shocks σ_γ is smaller on the second sample period, its credibility interval is much larger. There are a few possible reasons for this result. One of them is that the perception of risk might be really high over the second sample period, which may also be the cause for the violation the stylized fact that longer ends of the curve have smaller volatilities. Another possible reason is that the likelihood function of the second sample period is relatively flat for this parameter and, lastly, it might be the case that this parameter is not well identified[1].

[1]Although possible, we do not believe this is the case since the posterior for σ_γ on the

Table 4.1: Estimated Parameters

		Prior			Posterior - Sample 1			Posterior - Sample 2		
	Parameter Description	Distribution	Mean	St. Dev.	Mode	5%	95%	Mode	5%	95%
σ	Intertemporal elasticity of substitution	Gamma	2	1.4	2.9928	2.7939	3.3715	2.5517	2.1732	2.9880
$\bar{\gamma}$	Steady state of risk aversion	Gamma	10	3.2	9.4396	8.5033	11.0760	13.8117	11.3779	16.6632
φ	Disutility of labor	Gamma	1	0.44	1.6461	1.2771	2.5224	0.5762	0.3109	1.2883
α_H	Concavity of home production function	Beta	0.5	0.25	0.7925	0.7690	0.8104	0.3877	0.3602	0.4395
η	Degree of habit formation	Beta	0.7	0.17	0.6553	0.5611	0.7535	0.4557	0.3768	0.5447
τ	Degree of price indexation	Beta	0.8	0.15	0.6543	0.6323	0.7233	0.8941	0.7994	0.9159
ϕ_π	Inflation coefficient	Normal	3	1.5	4.9564	4.8987	5.0365	3.8394	3.6373	4.0607
ϕ_Y	Level of output coefficient	Gamma	0.5	0.22	0.3874	0.3627	0.4092	0.4363	0.3476	0.5161
ϕ_{dY}	Output growth coefficient	Gamma	0.5	0.22	0.8298	0.7435	0.9636	0.7728	0.5855	0.8901
ρ_R	Interest rate smoothing	Beta	0.7	0.1	0.9646	0.9597	0.9681	0.9376	0.9140	0.9460
ρ_π	Persistence of inflation target	Beta	0.95	0.3	0.9996	0.9985	0.9999	0.9989	0.9959	0.9996
σ_π	Volatility of inflation target shock	Inv. Gamma	0.01	0.03	0.0062	0.0058	0.0071	0.0057	0.0048	0.0065
ρ_v	Persistence of monetary policy shocks	Beta	0.5	0.25	0.6129	0.5855	0.6482	0.7550	0.7195	0.7895
σ_v	Volatility of monetary policy shocks	Inv. Gamma	0.01	0.03	0.0036	0.0033	0.0039	0.0035	0.0031	0.0043
ρ_γ	Persistence of risk aversion coefficient	Beta	0.9	0.1	0.9164	0.8881	0.9307	0.9844	0.9285	0.9902
σ_γ	Volatility of risk aversion shocks	Inv. Gamma	0.01	0.03	0.0393	0.0371	0.0416	0.0201	0.0117	0.1070
ρ_z	Persistence of the level of technology	Beta	0.9	0.1	0.9920	0.9881	0.9938	0.9732	0.9659	0.9753
σ_z	Volatility of technology shocks	Inv. Gamma	0.01	0.03	0.0287	0.0261	0.0318	0.0304	0.0278	0.0356

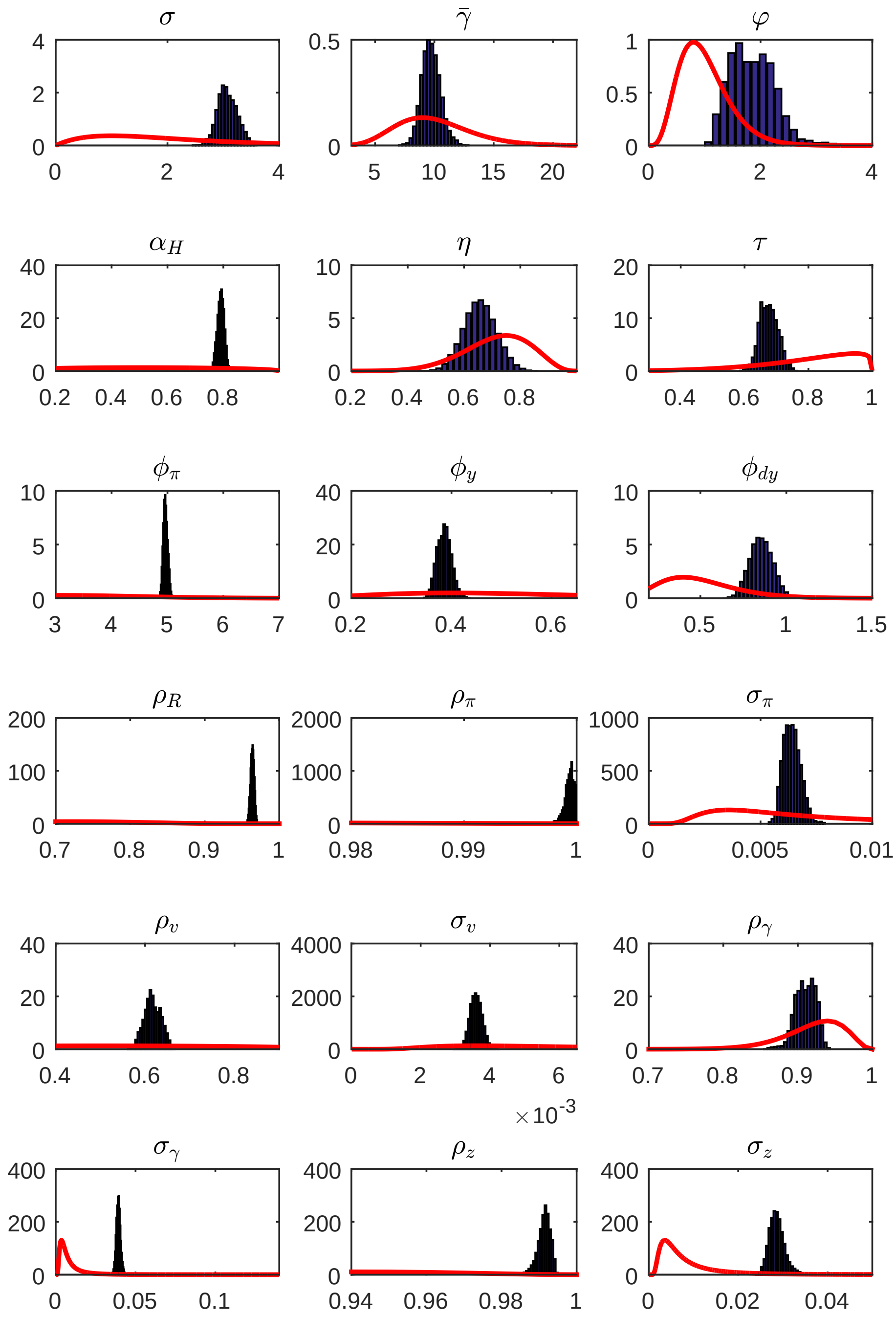

Figure 4.1: Sample 1 - Prior densities (red line) and sampled posterior densities (histogram).

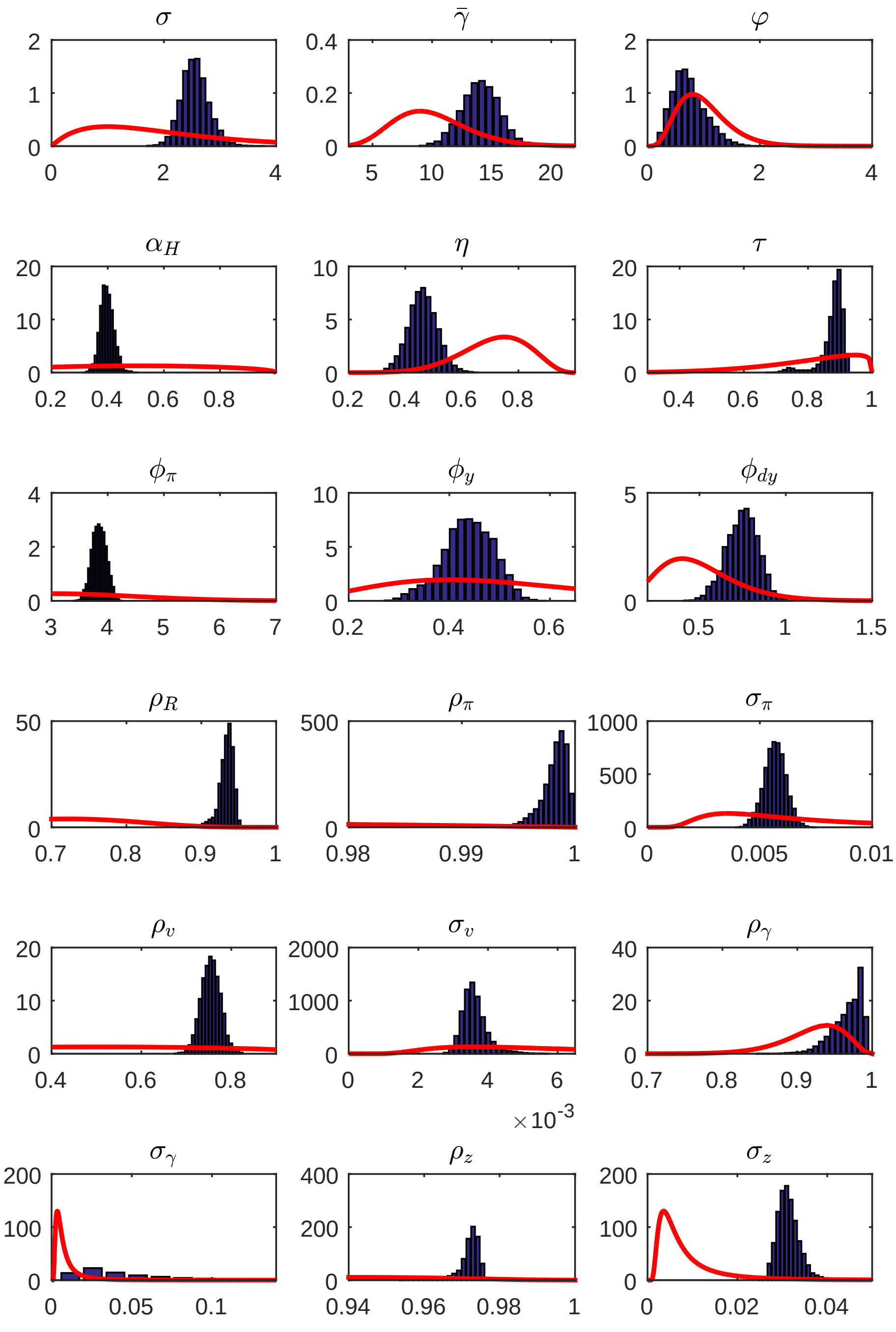

Figure 4.2: Sample 2 - Prior densities (red line) and sampled posterior densities (histogram).

Both output coefficients from the interest rate rule ϕ_Y and ϕ_{dY} have overlapping credibility for the two sample periods, so we can not say these ones have changed. The value of the inflation coefficient ϕ_π is smaller on the second period. If we make the assumption that the central bank choses its interest rate rule by minimizing a quadratic loss function with a term for inflation, a smaller inflation coefficient might be associated with a smaller weight for inflation in the loss function of the central bank[2]. Although quantitatively the estimates for the interest rate smoothing coefficient ρ_R are statistically different, there are both large and close enough to each other in order to say that, qualitatively, there has been no change. The estimated values for ρ_v are statistically different in both periods and higher for the second time frame. This increase in the persistence of monetary policy shocks should generate a hump-shaped impulse response function for the level of the yield curve, but since it was a small increase starting from a value that was not that low, there is no noticeable change in the shape of impulse response functions, as it can be seen in figure 4.4.

The persistence of the inflation target ρ_π is very close to 1 for both sample periods. Studies that also have time-varying inflation target usually calibrate this parameter to 0.99, although its estimated value from Castro et al (2011) is 0.83. Hordhal, Tristani and Vestin (2007) emphasize that a high value for this parameter is necessary for the model to generate a large term-spread. The volatility of inflation target shocks σ_π is also not statistically different between both periods and significantly smaller than the one from Castro et al (2011), which may be due to the fact that our model takes yield curve data into account and smaller values for shocks are expected when the model tries to capture higher order effects.

The posterior mean for the volatility of technology shocks σ_z are also smaller than the ones found in the literature and the credibility intervals for both sample periods overlap. The parameters for persistence of productivity shocks are statistically different but qualitatively the same for both sample periods and are in line with estimates from the literature.

In the essentially affine solution method, the parameters of shock volatilities have an effect over the steady-state values of the model. In the estimation process, instead of using the deviation from the mean we used the level of the yields in the observation equation. This keeps the kalman filter from choosing volatilities that are too high. This generates a trade-off between the fit of the term-structure, that requires lower estimates for the volatility parameters, and

first sample period has a very small standard deviation and mode very different from the prior mean.

[2]For a simple model, this result can be derived analytically, as shown in Walsh (2010).

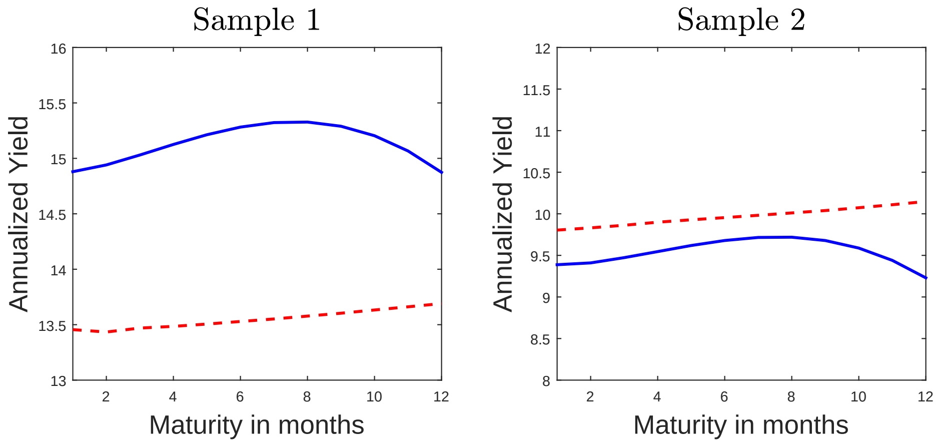

Figure 4.3: Steady-State of the yield curve. The solid line is the steady-state of the model and the dashed line is the empirical average.

the fit of the other observable variables, which require higher standard deviations of the shocks. Figure shows the steady-state generated by the estimated model and the empirical average of the term structure for both sample periods. We can see that the model does a better job in the second sample period. As Dew-Becker (2014) explains, the steady-state term spread can be interpreted as the average term-premium. Although the difference between the 12- and 1-month rate are small in these results, this is specific these set of estimated parameter values. With a different calibration, the model is capable of generating a highly positive term-spread but the trade-off on the fitting of the data makes this a difficult job in practice. The author of the solution method deals with this problem by adding a penalization term to the likelihood function, in order to force a better fit of the term structure by sacrificing a better fit of the macro variables.

Figure 4.4 shows the impulse response function for the term structure factors. These factor are constructed like Bekaert, Cho and Moreno (2010) and Dew-Becker (2014), where the level factor is defined as the average of all yields in the curve and the slope factor is the spread between the 12- and 1-month yield. The instantaneous impact of a monetary shock on the level of the curve has about half of the intensity on the second sample than it does on the first one. In both cases, the effect of the shock is not too persistent, lasting for about less than a year. The impact of the monetary policy shock over the slope of the curve is very similar on both sample periods. This did happen in our empirical VAR study, where although the shape of the responses of the slope to a monetary shock were similar, they were more intense on the second sample period. Given an inflation target shock, its effect on the level of the curve is strong and very persistent. It has an amplified effect on the longer ends of the curve so as to strongly increase the slope of the yield curve, although the effect

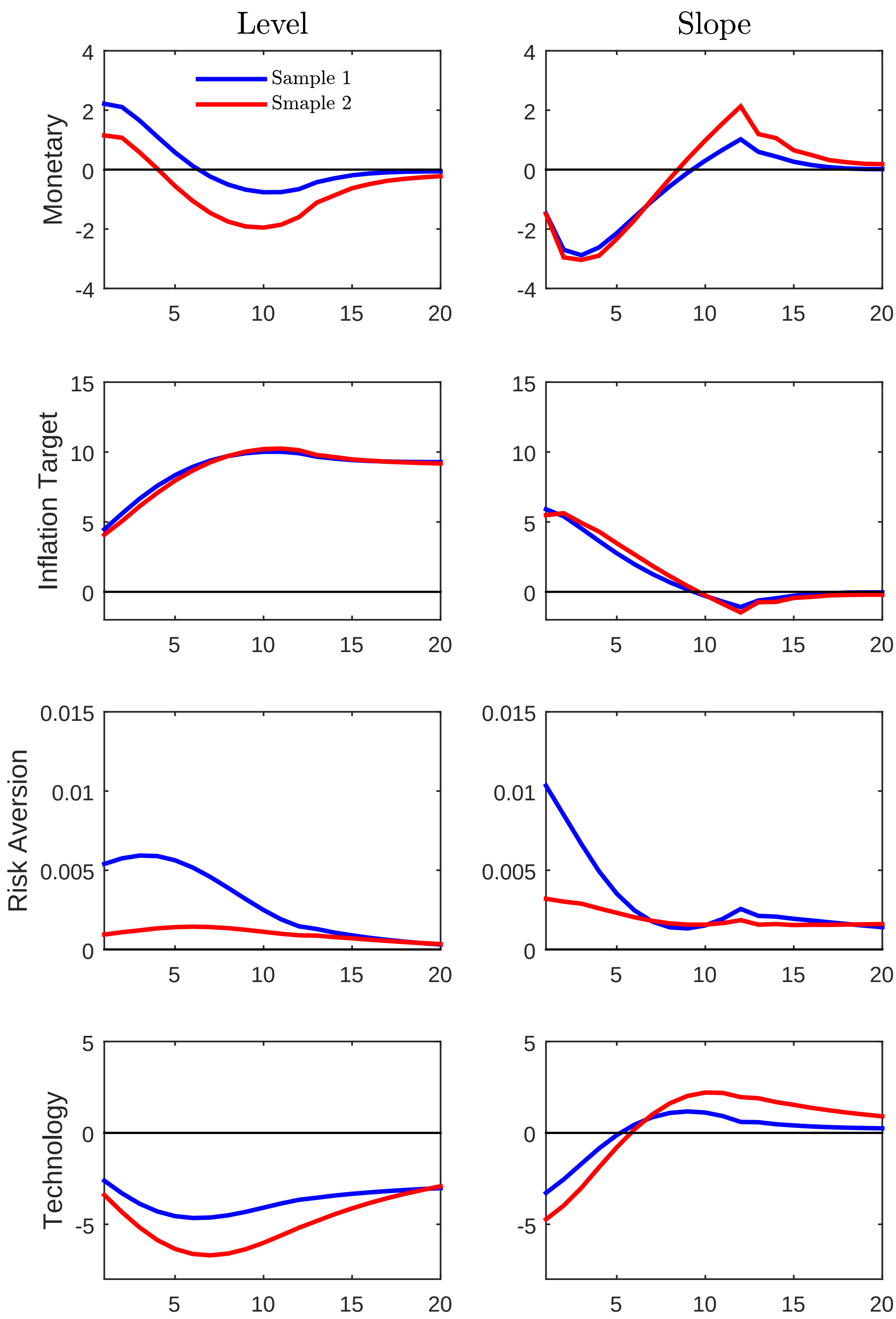

Figure 4.4: Impulse response function for the yield curve factors.

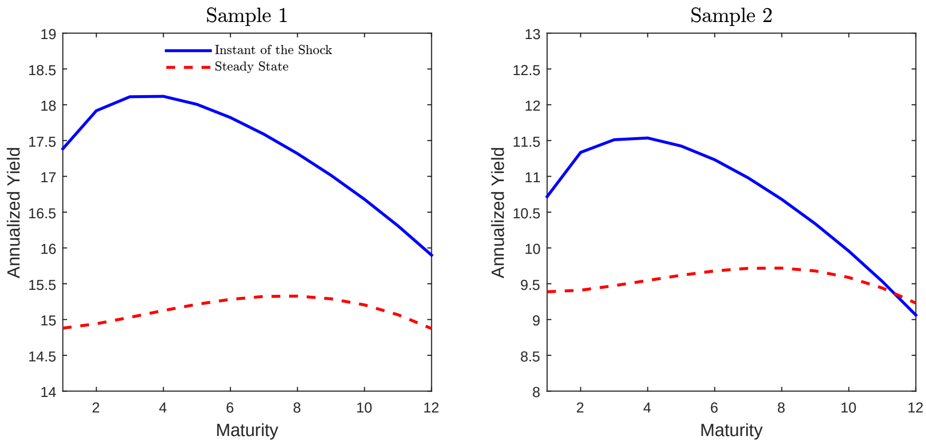

Figure 4.5: Yield curves on the instant of a monetary policy shock and their steady states

last only for about a year. Risk aversion shocks have a very small impact over both factors in both periods, but it is relatively stronger and more persistent in the first sample period and as expected, an increase in risk aversion leads to a higher yields and a higher spread. Technology shocks are also very persistent, generating long lasting effects over the term structure, specially over the level factor. Figure 4.5 shows the response of the whole yield curves for each sample period on the first instant of the monetary policy shock and their respective steady-states. We can see that the shock has a much stronger impact on the shorter yields and might have small negative effects on the longer yields. This is also found by Bekaert, Cho e Moreno (2010). Gürkaynk, Sack and Swanson (2005) also find this result in reduced form model but they depend on using several lags for their specification. While our model requires only one lag, it requires a very high persistence of the inflation target and interest rate smoothing to generate this result.

We can see from these results that monetary policy and risk aversion shocks are responsible for the short run movements of the term structure, while the central bank's inflation target and technology are responsible for the long run changes and there are no shocks with long lasting effects over the term spread. This can also be seen from a forecast error variance decomposition. We can see from table 4.2 that in the first sample, monetary shocks are initially responsible for 80% of the uncertainty of short rates, but only for 4.8% a year later. While inflation target and technology shocks are, together, initially responsible for 20% of the uncertainty, they add up to more than 95% of the long run uncertainty. Still in the first sample, the uncertainty of the longer ends of the curve are almost entirely due to inflation target and technology shocks. On the second sample period, inflation target shocks have a much higher importance on the volatility of short rates even on a small horizon. For

Table 4.2: Forecast error variance decomposition

Months ahead	Sample 1			
	ε_v	ε_π	ε_γ	ε_z
1-month rate				
1	80.0%	17.2%	0%	2.8%
3	48.2%	37.1%	0%	14.7%
6	16.7%	59.3%	0%	24.0%
12	4.8%	74.8%	0%	20.4%
12-month rate				
1	1.6%	76.7%	0%	21.7%
3	0.8%	78.4%	0%	20.8%
6	0.4%	80.6%	0%	19.0%
12	0.3%	83.9%	0%	15.8%
Months ahead	Sample 2			
	ε_v	ε_π	ε_γ	ε_z
1-month rate				
1	48.0%	45.0%	0%	6.9%
3	32.1%	38.6%	0%	29.2%
6	7.8%	49.5%	0%	42.7%
12	4.0%	61.1%	0%	34.9%
12-month rate				
1	0%	62.6%	0%	37.4%
3	0.3%	63.8%	0%	35.9%
6	0.9%	66.2%	0%	32.9%
12	1.1%	72.2%	0%	26.7%

a longer horizon inflation target and technology shocks still add up for more than 95% of the uncertainty, but not in the same proportions as in the first period. For the longer ends of the curve, monetary shocks have no importance in small horizons, but their share of uncertainty is higher for longer horizons. This might be a reflection of the non-occurring stylized fact of the Brazilian yield curve on the second sample period where longer maturities have a higher volatility than the short ones.

Another interesting benefit of this model formulation is that we can smooth the kalman filtered estimates of the non-observable variables of the system. Laubach and Williams (2003) extracts non-observable series from empirical models and Smets and Wouters (2003) do that from structural models. In our case, this is done using information from the term structure

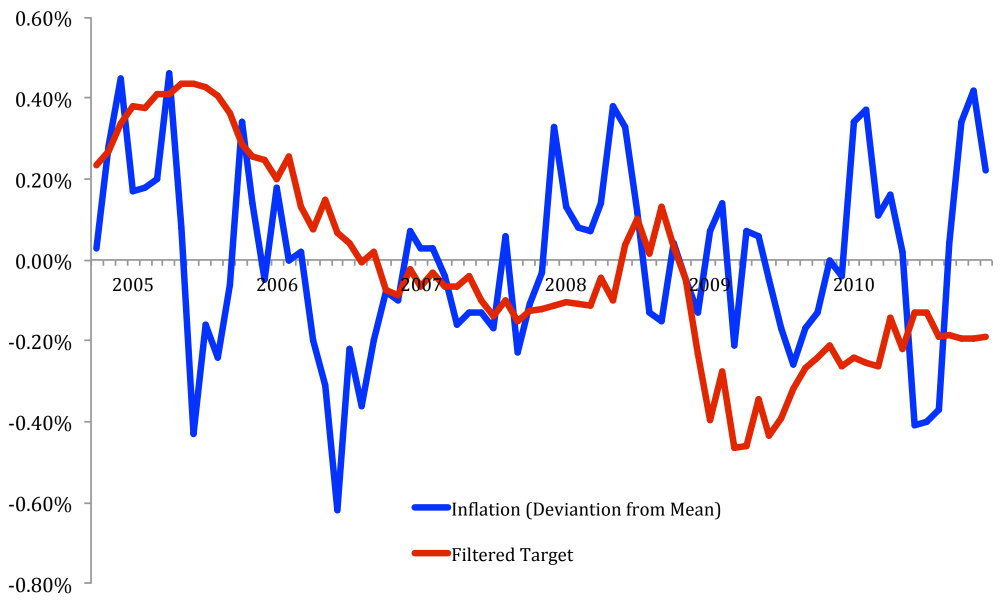

Figure 4.6: Monthly inflation rate and the smoothed filtered inflation target for the first sample period.

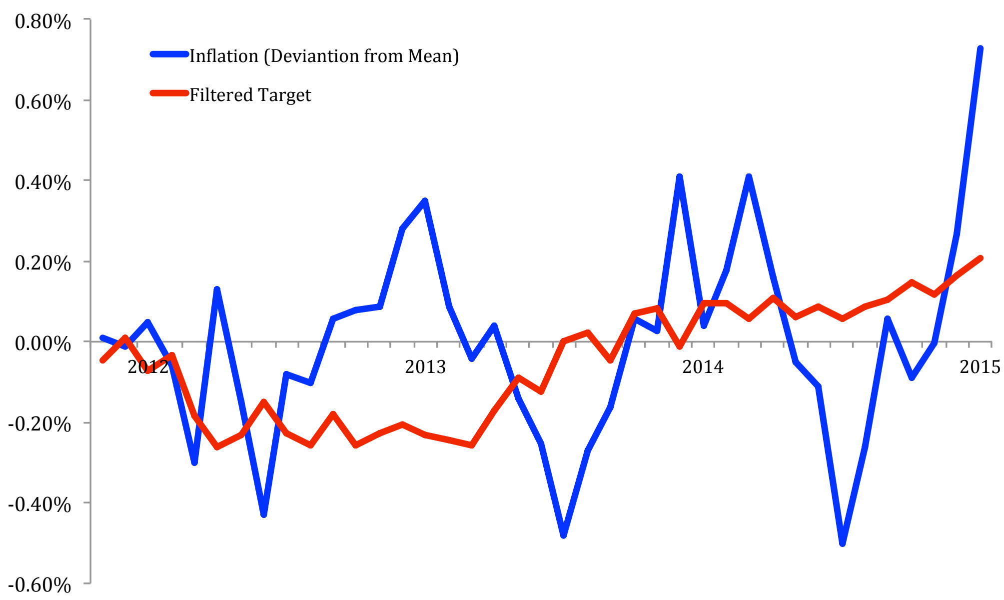

Figure 4.7: Monthly inflation rate and the smoothed filtered inflation target for the second sample period.

of interest rates. Bekaert, Cho and Moreno (2010) also extract the inflation target, natural interest rate and output gap from their model, but they do this analytically, and not by using the Kalman filter. Figures 4.6 and 4.7 show the smoothed filtered inflation target and the deviation from the sample mean of the inflation rate. For the first sample period, we can see that the inflation target has a downward trend, with the exception of the second semester of 2008. During this period, Brazilian inflation was always close to its official target and within the target band. For the second sample period, the extracted inflation target has an upward trend and it is less volatile than in the first period. We can see that from the middle of 2012 until the beginning of 2013, the inflation rate was higher than its extracted target and continuously increasing. During this same period the central bank's target rate was at its historical minimum. It was only at the end of 2013 that the extracted inflation target started to increase. From these results, its clear that the extracted inflation target should not be interpreted as the actual pursued target but one could look at it as how worried the central bank would be with the level of inflation, meaning that a lower extracted inflation target could mean that stabilizing inflation might not be the first objective. During the second half of the first sample period, the extracted inflation target was low, probably because the actual inflation also was. But the target remained low in the first half of the second sample period, even when actual inflation started increasing and interest rates were low.

5
Concluding Remarks

This paper studies the empirical and structural relationship between monetary policy shocks and the Brazilian yield curve. We analyze the empirical relationship of the central bank's target rate and the impact of monetary shocks over the yield curve for two sample time frames using reduced form VARs and compare the result between some emerging market inflation targeting countries. We find that there has been a change in the way that the Brazilian yield curve responds to these monetary shocks. We then present risk adjusted solution methods for DSGE models and use the Essentially Affine method from Dew-Becker (2012) in order to estimate a DSGE model with a yield curve, Epstein-Zin preferences and time-varying risk-aversion to see if a structural model can capture these changes. We find that some changes are captured, such as the increased volatility of yields for longer maturities and statistically significant changes in the values of structural parameters, but the model can not capture changes in the shape of the impulse response function.

The contribution of this study is to show that additional observable data from the yield curve can help shed some light on structural changes by bringing more information to the estimation procedure. The change in the estimated value of parameters for each sample period and the extracted inflation target both point toward a central bank that has not been very worried with the level of inflation over the second time frame.

6
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7
Appendix

A: Avaiability of yield curve data

In section 2 of this paper we used macroeconomic data from six countries: Brazil, Chile, Colombia, Mexico, South Africa and Turkey. We split the series in two sample periods. The first one is from January of 2004 until December of 2010 and teh second is from January 2011 until January 2015. Not all the series for yields were long enough to allow us to use all countries in both sample periods of our analysis. Antoher issue was that the series of interest rates for mexico showed had the presence of a unit root over the second sample period, which would compromise the stability of our estimated VAR model, so we removed Mexico from the second sample period, but kept it in the first one. The starting dates for the series of yields are show in table 7.1.

Table 7.1: Summary of avaiable data

	Starting date of shortest series	Used in sample period
Brazil	Jan 2004	1 and 2
Chile	Mar 2010	2
Colombia	Feb 2011	2
Mexico	Jan 2004	1
South Africa	Jan 2004	1 and 2
Turkey	May 2010	2